<u>**ONLINE COURSE WARE**</u>

SUBJECT NAME: MATHEMATICS III

SUBJECT CODE: M 301

TOTAL NO. OF LECTURES: 44

CONTACT HOURS: 44 HOURS.

CREDIT: 4

Target Streams: ECE, AEIE, ME

Module No	Lecture No	Topic
MODULE I: Fourier Series and Fourier Transform	1	Introduction, Periodic functions: Properties, Even & Odd functions: Properties,
	2	Special wave forms: Square wave, Half wave Rectifier, Full wave Rectifier, Saw-toothed wave, Triangular wave. Euler's Formulae for Fourier Series,
	3	Fourier Series for functions of period 2π, Fourier Series for functions of period
	4	Dirichlet's conditions, Sum of Fourier series. Examples.
	5	Theorem for the convergence of Fourier Series Fourier Series of a function with its periodic extension. Half Range Fourier Series: Construction of Half range Sine Series,
	6	Construction of Half range Cosine Series.
	7	Parseval's identity (statement only).Examples
	8	Fourier Integral Theorem, Fourier Transform of a function, Fourier Sine and Cosine Integral Theorem with examples. Fourier Cosine & Sine Transforms.
	9	Transform of some standard function**,** Fourier, Fourier Cosine & Sine Transforms of elementary functions, Properties of Fourier Transform. : Linearity, Shifting, Change of scale, Modulation
	10	Fourier Transform of Derivatives, Examples. Convolution Theorem with examples.
	11	verse Fourier transform or Fourier Integral, Fourier Sine and Cosine Integral Theorem Inverse Property of Inverse Fourier Transform.
	12	Some examples with solution of Fourier Transform, Exercise.
MODULE II: Probability	13	Basic concepts of Random Variable and its Probability Distribution
	14	Discrete Probability Distribution
	15	Expectation, Variance & S.D. of Discrete Random Variables
	16	Continuous Probability Distribution
	17	Expectation, Variance & S.D. of Continuous Random Variables
	18	Binomial Distribution
	19	Poisson Distribution
	20	Normal and Standard Normal Distribution

Distributions		
	21	Transformation of a Normal Distribution into Standard Normal Distribution
	22	Binomial Approximation to Poisson and Normal Distribution
MODULE III: Calculus of Complex Variable	23	Introduction to Functions of a Complex Variable, Concept of Limit, Continuity
	24	Concept of Differentiability. Analytic functions
	25	Cauchy-Riemann Equations (statement only). Sufficient condition for a function to be analytic, Laplace Equation Harmonic function and Conjugate Harmonic function.
	26	Introduction to Complex Integration Concept of simple curve, closed curve, smooth curve & contour. Cauchy's theorem (statement only).
	27	Cauchy-Goursat theorem (statement only). Line integrals along a piecewise smooth curve. Examples
	28	Cauchy's integral formula
	29	Cauchy's integral formula for the derivative of an analytic function, Cauchy's integral formula for the successive derivatives of an analytic function. Examples
	30	Taylor's series, Laurent's series. Examples
	31	Zero of an Analytic function, order of zero, Singularities of an analytic function. Isolated and non-isolated singularity, essential singularities. Poles: simple pole, pole of order m
	32	Examples on determination of singularities and their nature.
	33	Residue
	34	Cauchy's Residue theorem (statement only). problems on finding the residue of a given function
MODULE IV: Partial Differential Equation (PDE) & Series	35	Basic Concepts of PDE
	36	Separation of Variable Method
	37	One-Dimensional Wave Equation

Solution Of Ordinary Differential Equation (ODE)	38	One-Dimensional Heat Equation
	39	Two-Dimensional Laplace Equation
	40	Power Series - Properties
	41	Series Solution of ODE-I
	42	Series Solution of ODE-II
	43	Bessel's Function
	44	Legendre Polynomial

MODULE I

Fourier Series and Fourier Transform

Lecture 1:

Introduction to Fourier Series

There are many types of series expansions for functions. The Maclaurin series, Taylor series, Laurent series are some such expansions. But these expansions become valid under certain strong assumptions on the functions (those assumptions ensure convergence of the series). Fourier series also express a function as a series and the conditions required are fairly good and suitable when we deal with signals.

Suppose f is a real valued function from R to R. In this note, we deal with the following three questions:

- When does *f* has a Fourier seriesexpansion?
- How we find the expansion?
- What are the main properties of thisexpansion?

Existence of a Fourier series expansion:

There are three conditions which guarantees the existence of valid Fourier expansion for a given function. These conditions are collectively called **Dirichlet conditions:**

1. f is a periodic function on R.This means that there exists a period $T \geq 0$ such that

$$f(x)=f(x+T) \text{ for all } x \in \mathrm{R}.$$

2. f has only a finite number of maxima and minima in aperiod.

3 f has atmost a finite number of discontinuous points inside aperiod.

4 f is integrable over the period of the function.

It should be noted that the second and third conditions are satisfied by many real valued functions that we deal with, inside any finite interval. But periodicity is a condition that is satisfied by very few functions,for example, *constant function*, *sine, cos, tan* and their combinations. But we can consider any function defined on a finite interval $[a,b]$ (or(a,b)) as a periodic function on R by thinking that the function is extended to R by repeating the values in$[a,b]$to the remaining part of R.

Even & Odd functions and their Properties:

A function f(x)is said to be even function if f(-x)=f(x) for all values of x; e.g. the functions cos(x), x^2 all are even functions.

A function f(x)is said to be an odd function if f(-x)=-f(x) for all values of x; e.g. the functions sin(x), x^3 all are odd functions.

Graph of an even function: One of the most important properties of even functions is

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

Graph of an odd function: One of the most important properties of even functions is

$$\int_{-a}^{a} f(x)dx = 0$$

Lecture 2:

Special wave forms:

The graph of every periodic functions runs like a wave-this is wave-form. Below we show some typical wave-form which are usually met in communication engineering:

(i) Square waveform: Consider the periodic function f(x) defined by

$f(x)=-k, \quad -a<x<0$

$\quad =k, \quad 0<x \le a$

And $f(x+2a)=f(x)$ for all x

This kind of graph is known as Square Waveform.

(ii) Half wave Rectifier: Consider the periodic function f(x) defined by

$f(x)=-k\sin x, \quad 0\le x\le \pi$

$\quad =0, \quad \pi \le x \le 2\pi$

And $f(x+2\pi)=f(x)$ for all x

This kind of graph is known as Half wave Rectifier.

(iii) Full wave Rectifier: Consider the periodic function f(x) defined by

$f(x)=-k\sin x, \quad 0\le x\le \pi$

And $f(x+\pi)=f(x)$ for all values of x

This kind of graph is known as Full wave Rectifier.

(iv) Saw-toothed wave: Consider the periodic function f(x) defined by

$f(x)=x, \quad -a\le x\le a$

And $f(x+2a)=f(x)$ for all values of x

This kind of graph is known as Saw-toothed Waveform.

(v) Triangular wave: Consider the periodic function f(x) defined by

$$f(x)=1+\frac{2x}{a}, \quad -a<x\le 0$$

$$=1-\frac{2x}{a}, \quad 0\le x\le a$$

And $f(x+2a)=f(x)$ for all values of x

This kind of graph is known as Triangular Waveform.

Examples:

1. Extend the function $f(x)=0, \ -3<x<0$

$=x^2, \ 0<x<3$

to a periodic function.

Sol. This function is defined on the interval (-3, 3) only. To extend in periodic form just define the function on $(-\infty,\infty)$ by the rule $f(x+6)=f(x)$ for all values of x.

Euler's Formulae for Fourier Series:

Let f(x) be defined and integrable in the interval (-T,T).

Extend the function to a periodic function of period 2T by defining f(x+2T)=f(x) for all values of x of period 2T. The Fourier series of f(x) is given by

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{T}+b_n\sin\frac{n\pi x}{T})$$

Where a_n, b_n are called Fourier co-efficient and these are, according to Euler,

$$a_0=\frac{1}{T}\int_{-T}^{T}f(x)dx$$

$$a_n=\frac{1}{T}\int_{-T}^{T}f(x)\cos\frac{n\pi x}{T}dx$$

$$b_n=\frac{1}{T}\int_{-T}^{T}f(x)\sin\frac{n\pi x}{T}dx$$

Where n=1,2,3,…….

1. **Examples:** Consider the function f(x)=3, $0<x\le 5$
   =-3, $-5<x\le 0$

We extend the function by defining f(x+10)=f(x) for all x. so this becomes a periodic function of period 10. This gives a square waveform.

Sol. The Fourier co-efficient, according to Euler Formula, are

$$a_0=\frac{1}{5}\int_{-5}^{5}f(x)dx$$

$$=\frac{1}{5}\left\{-3\int_{-5}^{0}dx+3\int_{0}^{5}dx\right\}=0$$

$$a_n=\frac{1}{5}\int_{-5}^{5}f(x)\cos\frac{n\pi x}{5}dx$$

$$=\frac{1}{5}\left\{-3\int_{-5}^{0}\cos\frac{n\pi x}{5}dx+\int_{0}^{5}f(x)\cos\frac{n\pi x}{5}dx\right\}$$

$$=\frac{1}{5}\left\{-3\int_{0}^{5}\cos\frac{n\pi x}{5}dx+\int_{0}^{5}f(x)\cos\frac{n\pi x}{5}dx\right\}=0$$

And $b_n=\frac{1}{5}\int_{-5}^{5}f(x)\sin\frac{n\pi x}{5}dx$

$$= \frac{2}{5}\int_0^5 f(x)\sin\frac{n\pi x}{5}dx$$

$$= \frac{6}{5}\int_0^5 \sin\frac{n\pi x}{5}dx$$

$$= -\frac{6}{5}\left[\frac{\cos\frac{n\pi x}{5}}{\frac{n\pi}{5}}\right]_0^5 = -\frac{6}{n\pi}(\cos n\pi - 1) = \frac{6(1-\cos n\pi)}{n\pi}$$

Therefore the Fourier series of f(x) is

$$\frac{0}{2} + \sum_{n=1}^{\infty}(0.\cos\frac{n\pi x}{5} + \frac{6(1-\cos n\pi)}{n\pi}\sin\frac{n\pi x}{T})$$

$$= \sum_{n=1}^{\infty}\frac{6}{\pi}.\frac{(1-\cos n\pi)}{n}\sin\frac{n\pi x}{5}$$

$$= \frac{6}{\pi}\left\{(1-\cos\pi)\sin\frac{\pi x}{5} + \frac{(1-\cos 2\pi)}{2}\sin\frac{2\pi x}{5} + \frac{(1-\cos 3\pi)}{3}\sin\frac{3\pi x}{5} + \text{..........}\right\}$$

We see f(0)=-3 but the values of the Fourier series at x=0 is $\frac{6}{\pi}\{0+0+0+......\} = 0$

Lecture 3:

Fourier Series for functions of period 2π:

The above Fourier series for T= π i.e. the Fourier series for the function f(x) defined and integrable on (- π , π) and f(x+2 π)=f(x) for all values of x is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

Where the Fourier co-efficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$,for n=1,2,3,……

Examples: Expand f(x)=x in Fourier Series on the interval $-\pi \le x \le \pi$.

Sol. Observe that f(x)=x is bounded and integrable on $-\pi \le x \le \pi$,since it is continuous there. Further

$f'(x) = 1 > 0$ indicates that f(x) is monotone increasing on the entire interval. We extend this by defining f(x+2π)=f(x) for all values of x. This is a periodic function of period 2π. Its Fourier series corresponding to f(x) is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Where the Fourier co-efficient are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx dx$$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x dx = 0$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx dx = 0$

Since $x \cos nx$ and x are odd functions ,and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx dx = \frac{2}{\pi}\int_{0}^{\pi} x \sin nx dx$$

Since $x \sin nx$ is even. Thus

$$b_n = \frac{2}{\pi}\left[-x\frac{\cos nx}{n}\right]_0^{\pi} + \frac{2}{\pi}\int_0^{\pi}\frac{\cos nx}{n}dx$$

$$= -\frac{2}{n}\cos n\pi = \begin{cases} -\frac{2}{n}, & n \quad \text{even} \\ \frac{2}{n}, & n \quad \text{odd} \end{cases}$$

Hence f(x)=x generates Fourier Series in the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots\ldots..$$

$$= 2\left\{\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\ldots.\right\}$$

Lecture 4:

Dirichlet's conditions:

A function f(x) will be said to satisfy Dirichlet's condition on an interval $-\pi \le x \le \pi$ in
Which it is defined when it is subjected to one of the two two following conditions:
(i)f(x) is bounded in [-T,T] and the interval [-T,T] can be decomposed in a finite number of sub-intervals such that f(x) is monotonic (increasing or decreasing) on each of the sub-intervals.
(ii) f(x) has a finite number of points of infinite discontinuity in [-T,T].
When arbitrary small neighbourhoodof these points are excluded from [-T,T] f(x) becomes bounded in the remaining part and this remaining part can be decomposed into a finite number of sub-intervals such that f(x) is monotonic in each of the sub-intervals. Moreover the improper integral $\int_{-\pi}^{\pi} f(x)dx$ is absolutely convergent.

1. **Convergence:**

When f(x) satisfies Dirichlet's condition on $-\pi \leq x \leq \pi$, the Fourier Series corresponding to f(x) converges to f(x) at any point x on $-\pi \leq x \leq \pi$ when f(x) is continuous and converges to $\frac{1}{2}\{f(x+0)+f(x-0)\}$ when there is an ordinary discontinuityat the point. In particular at $x=\pi$ and $x=\pi$ it converges to $\frac{1}{2}\{f(-\pi+0)+f(\pi-0)\}$ when $f(-\pi+0)$ and $f(\pi-0)$ exist.

Example: Let f(x)= x-3 , $-3 \leq x \leq 0$

=3-x, $0 < x \leq 3$.

f(x) is bounded in [-3,3]. The interval[-3,3] is decomposed as [-3,0] $\cup$ [0,3] such that f(x) is increasing in [-3,0] and decreasing in [0,3]. So we conclude this function f(x) satisfies Dirichlet's condition.

Lecture 5:

Fourier Series of a function with its periodic extension:

We introduce Fourier Series of a function f(x) which is primarily defined on the interval [-T,T] and then extending it to a periodic wave. But the function may appear as defined primarily on an interval [c,c+2T] where c may be any real number. In that case also we have no trouble of getting its Fourier Series.

Theorem: If f(x) be defined and integrable in [c,c+2T] and f(x+2T)=f(x) for all values of x , then the Fourier Series of f(x) is also

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{T}+b_n \sin\frac{n\pi x}{T})$$

Where the Fourier co-efficients are

Where c may be anyreal number.

Half Range Fourier Series:

A trigonometric series like the fourier Series is called a Half Range Fourier Series if only sine terms or only cosine terms are present. When only sine terms are present the series is called Half Range Sine Series; when only cosine terms are present the series is called Half Range Cosine Series.

When a half range series corresponding to a function is desired, the function is generally defined in the interval (0, T) which is half of the interval (-T, T).

Construction of Half range Sine Series:

Let f(x)be a function defined and integrable on the interval (0, T). We extend the domain of definition to [-T,0] defining by f(-x)=-f(x). This extension is shown in the adjacent figure. Then this extended f(x) becomes odd in the interval [-T, T].

$$a_0 = \frac{1}{T}\int_{-T}^{T} f(x)dx$$

$$= 0 \qquad [\because f(x) \text{ is odd.}]$$

$$a_n = \frac{1}{T}\int_{-T}^{T} f(x)\cos\frac{n\pi x}{T}dx$$

$$= 0 \quad [\therefore f(x)\cos\frac{n\pi x}{T} \text{ is an odd function}]$$

$$b_n = \frac{1}{T}\int_{-T}^{T} f(x)\sin\frac{n\pi x}{T}dx$$

$$= \frac{2}{T}\int_{0}^{T} f(x)\sin\frac{n\pi x}{T}dx \quad [\therefore f(x)\sin\frac{n\pi x}{T} \text{ is an even function}].$$

The Fourier Series of f(x) becomes

$$\frac{0}{2} + \sum_{n=1}^{\infty}(0.\cos\frac{n\pi x}{T} + b_n \sin\frac{n\pi x}{T})$$

i.e. $\sum_{n=1}^{\infty}(b_n \sin\frac{n\pi x}{T})$

which is the required Half Range Sine Series. Obviously if f(x) satisfies Dirichlets condition in [0, T] then this series is convergent and the value is as for Fourier Series.

Lecture 6:

Construction of Half range Cosine Series:

Let f(x)be a function defined and integrable on the interval (0, T). We extend the domain of definition to [-T,0] defining by f(-x)=f(x). This extension is shown in the adjacent figure. Then this extended f(x) becomes odd in the interval [-T, T].

$$a_0 = \frac{1}{T}\int_{-T}^{T} f(x)dx = \frac{2}{T}\int_{0}^{T} f(x)dx \quad [\because f(x) \text{ is even.}]$$

$$a_n = \frac{1}{T}\int_{-T}^{T} f(x)\cos\frac{n\pi x}{T}dx$$

$$= \frac{2}{T}\int_{0}^{T} f(x)\cos\frac{n\pi x}{T}dx \quad [\because f(x)\cos\frac{n\pi x}{T} \text{ is an even function}]$$

$$b_n = \frac{1}{T}\int_{-T}^{T} f(x)\sin\frac{n\pi x}{T}dx = 0 \quad [\because f(x)\sin\frac{n\pi x}{T} \text{ is an odd function}]$$

Consequently the Fourier Series of f(x) becomes

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{T} + 0.\sin\frac{n\pi x}{T})$$

i.e. $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{T}$.

Which is the required Half Range Cosine Series.

Here also this series converges according as f(x) satisfies Dirichlet's Condition.

In particular if the interval of definition becomes $[0,\pi]$ then

(1) The half Range Sine series becomes $\sum_{n=1}^{\infty}(b_n \sin nx)$ **where**

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx dx$$

(2) The half Range Cosine series becomes $\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx$ **where** $a_0=\frac{2}{\pi}\int_0^{\pi} f(x)dx$ **and** $a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nxdx$ **.**

Example: Consider the function f(x)=sinx , $0 < x \le \pi$. To get its Half Range Cosine series we extend the function to the interval ($-\pi$,0) defining by f(-x)=f(x). With this extension f(x) becomes even on the interval ($-\pi,\pi$). Then

$$a_0=\frac{2}{\pi}\int_0^{\pi}\sin dx=\frac{4}{\pi}.$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}\sin x\cos nxdx \qquad (1)$$

$$=\frac{1}{\pi}\int_0^{\pi}\{\sin(x+nx)+\sin(x-nx)\}dx$$

$$=\frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right]_0^{\pi} \quad \text{for } n\neq 1$$

$$=\frac{-2(1+\cos n\pi)}{\pi(n^2-1)}=\frac{2(1+\cos n\pi)}{\pi(1-n^2)} \quad \text{for } n\neq 1$$

Now $a_1=0$ (from (1)) and $b_n=0$.

Therefore, the Half Range Cosine series becomes

$$\frac{1}{2}.\frac{4}{\pi}\sum_{n=2}^{\infty}\left(\frac{2(1+\cos n\pi)}{\pi(1-n^2)}\cos nx+0.\sin nx\right)$$

$$=\frac{2}{\pi}-\frac{2}{\pi}\sum_{n=2}^{\infty}\left(\frac{2(1+\cos n\pi)}{\pi(1-n^2)}\cos nx\right)$$

$$=\frac{2}{\pi}-\frac{4}{\pi}\left(\frac{\cos 2x}{2^2-1}+\frac{\cos 4x}{4^2-1}+\frac{\cos 6x}{6^2-1}+\ldots\ldots\ldots\right)$$

Lecture 7:

Parseval's Identity

If the Fourier Series of a function f(x) converges uniformly to f(x) in the interval (-T,T) then $\frac{1}{T}\int_{-T}^{T}\{f(x)\}^2 dx=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a^2{}_n+b^2{}_n)$

Where a_n, b_n are Fourier Co-efficients of f(x).

Note: 1) Corresponding to Half Range sine series $\sum_{n=1}^{\infty} b_n \sin nx$ the Persaval'sindentity would be $\frac{2}{T}\int_0^T \{f(x)\}^2 dx = \sum_{n=1}^{\infty} b^2{}_n$ since here f(x) is extended to an even function.

2) Corresponding to Half Range cosine series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ the Persaval's indentity would be $\frac{2}{T}\int_0^T \{f(x)\}^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$ since here f(x) is extended to an odd function i.e. $\{f(x)\}^2$ is extended to even.

Example: Consider the function f(x)= -x, $-2 < x \le 0$

$= x$, $0 \le x \le 2$.

Find the Fourier series of this function.

Solution: We see f(x) is an even function. Extending this to a periodic function defined by f(x+4)=f(x).

Here $a_0 = \frac{1}{2}\int_{-2}^{2} f(x)dx = \frac{1}{2}\int_{-2}^{0} -xdx + \frac{1}{2}\int_{0}^{2} xdx = 2$

$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx = \frac{2}{2}\int_{0}^{2} x\cos\frac{n\pi x}{2}dx$ [since $f(x)\cos\frac{n\pi x}{2}$ is an even function].

$= \left[x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - 1\left(\frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right]_0^2$

$= \frac{4}{n^2\pi^2}(\cos n\pi - 1)$ for $n \ne 0$.

$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}dx = 0$ [since $f(x)\sin\frac{n\pi x}{2}$ is an odd function]

Again the function f(x) satisfies Dirichet's condition and it is continuous everywhere

So its Parseval's Identity is

$$\frac{1}{2}\int_{-2}^{2} \{f(x)\}^2 dx = \frac{2^2}{2} + = \sum_{n=1}^{\infty}\left[\left\{\frac{4}{n^2\pi^2}.(\cos n\pi - 1)\right\}^2 + 0^2\right]$$

Or, $\frac{1}{2}\left[\int_{-2}^{0}(-x)^2\,dx+\int_{0}^{2}x^2dx\right]=2+\sum_{n=1}^{\infty}\frac{16}{n^4\pi^4}.(\cos n\pi-1)^2$

Or, $\frac{1}{2}\left[\frac{x^3}{3}\right]_{-2}^{2}=2+\frac{64}{\pi^2}\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+..............\right)$

Or, $\frac{8}{3}=2+\frac{64}{\pi^2}\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+..............\right)$

Or, $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+..............=\frac{\pi^4}{96}$

Lecture 8.

Introduction to Fourier Transform

In the previous chapter we have seen if a function f(x) satisfies dirichlel's condition on the interval (-T, T) then it can be expanding in the Fourier Series

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{T}+b_n\sin\frac{n\pi x}{T})$$

Where a_n, b_n are Fourier coefficients. Using the Euler's identity $e^{i\theta}=\cos\theta+i\sin\theta$ the Fourier Series of f(x) can be written as

$$f(x)=\sum_{n=1}^{\infty}c_n e^{\frac{in\pi x}{T}} \qquad (1)$$

Where $c_n=\frac{1}{2T}\int_{-T}^{T}f(x)e^{-\frac{in\pi x}{T}}dx$ are also known as fourier coefficients. If now this hold for all values of T as $T\to\infty$, the expansion (1) takes the form f(x)= $=\int_{-\infty}^{\infty}\left\{\frac{1}{2\pi}F(s)\right\}e^{-isx}ds$

Where F(s)= $=\int_{-\infty}^{\infty}f(t)e^{ist}dt$ which is known as Fourier Transform.

Fourier Transforms of a function:

It transforms an integrable function to an another function defined as follows

Let f(x) be integrable function on any interval (-T, T). Then the improper integral

$$\Im(f(x)) = F(s) = \int_{-\infty}^{\infty} f(x)e^{isx}dx$$

is called the Fourier Transform of the function f. This is a function of s.

Fourier Sine and Cosine Integral Theorem:

The Fourier sine transform of f(x) , $0 < x < \infty$ is defined as

$$\Im(f(x)) = F_s(s) = \int_{-\infty}^{\infty} f(x)\sin(sx)dx$$

The Fourier Cosine transform of f(x) , $0 < x < \infty$ is defined as

$$\Im(f(x)) = F_c(s) = \int_{-\infty}^{\infty} f(x)\cos(sx)dx$$

provided the improper integral are convergent.

Illustration:

(i) Let us consider the function

f(x)=1 , $-1 \le x \le 1$

=0 , otherwise

It's Fourier Transform,

$$F(s) = \int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$= \int_{-\infty}^{-1} 0.e^{isx}dx + \int_{-1}^{1} 1.e^{isx}dx + \int_{1}^{\infty} 0.e^{isx}dx = \int_{-1}^{1} 1.e^{isx}dx$$

$$= \left[\frac{e^{isx}}{is}\right]_{-1}^{1} = \frac{e^{is} - e^{-is}}{is} = \frac{(\cos s + i\sin s) - (\cos s - i\sin s)}{is}$$

$$= \frac{2\sin s}{s} \quad \text{for} \quad s \neq 0.$$

For, s=0, $F(0) = \int_{-1}^{1} 1.e^{0.x}dx = \int_{-1}^{1} dx = 2$

So, the Fourier Transform of f(x) is

$$F(s) = \frac{2\sin s}{s}, \quad s \neq 0$$

$$=2, \quad s=0$$

Fourier Cosine & Sine Transforms:

1. If f(x), $-\infty < x < \infty$ is an even function then its Fourier Transform $F(s) = 2F_c(s)$ where $F_c(s)$ is the Fourier cosine transform of f(x).

Proof: $F(s) = \int_{-\infty}^{\infty} f(x)e^{isx}dx$

$$= \int_{-\infty}^{\infty} f(x)(\cos sx + i \sin sx)dx$$

$$= \int_{-\infty}^{\infty} f(x)\cos sxdx + i\int_{-\infty}^{\infty} f(x)\sin sxdx$$

$$= 2\int_{0}^{\infty} f(x)\cos sxdx + i.0$$

[since $f(x)\cos sx$ is an odd function and f(x)sinsx is odd]

2. If f(x), $-\infty < x < \infty$ is an odd function then its Fourier Transform $F(s) = 2iF_s(s)$ where $F_s(s)$ is the Fourier sine transform of f(x).

Illustration:

Let us consider the function f(x)=1 , -1<x<1

=0 , otherwise.

is an even function. So its Fourier Transform, $F(s) = 2F_c(s)$

or, $F_c(s) = \frac{1}{2}F(s)$

So, its Fourier Cosine Transform,

$$F_c(s) = \frac{1}{2}.\frac{2\sin s}{s}, \quad s \neq 0$$

$$= \frac{1}{2}.2, \quad s=0$$

i.e. $F_c(s) = \frac{\sin s}{s}, \quad s \neq 0$

=1, s=0.

Lecture 9.

Fourier Cosine & Sine Transforms of elementary functions:

Transform of some standard function:

The Fourier transform are

(1) $F(s) = \sqrt{2\pi}e^{-\frac{s^2}{2}}$ of $f(x) = e^{-\frac{x^2}{2}}$

(2) $F(s) = \frac{2a}{a^2 + s^2}$ of $f(x) = e^{-a|x|}$

(3) $F(s) = \frac{2\sin sa}{s}, \quad s \neq 0$ of f(x)=1, $|x| < a$

=2a , s=0 =0 , $|x| > a$

(4) $F_s(s) = \dfrac{s}{a^2 + s^2}$ of $f(x) = e^{-ax} \, (a > 0)$

(5) $F_c(s) = \dfrac{a}{a^2 + s^2}$ of $f(x) = e^{-ax} \, (a > 0)$

Proofs.

(1) The Fourier Transform corresponding to f(x) is

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{isx} dx = \int_{-\infty}^{\infty} e^{\frac{-x^2+2isx}{2}} dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{x^2-2isx}{2}} dx = \int_{-\infty}^{\infty} e^{-\frac{x^2-2isx+(is)^2+s^2}{2}} dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{(x-is)^2+s^2}{2}} dx = e^{-\frac{s^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(x-is)^2}{2}} dx$$

[putting t=x-i.s i.e. dt=dx]

$$= e^{-\frac{s^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt = 2e^{-\frac{s^2}{2}} \int_{0}^{\infty} e^{-\frac{t^2}{2}} dt$$

$$= 2\sqrt{2} e^{-\frac{s^2}{2}} \int_{0}^{\infty} e^{-z^2} dz$$ [putting $\dfrac{t}{\sqrt{2}} = z,$ i.e. $\dfrac{1}{\sqrt{2}} dt = dz,$]

$$= 2\sqrt{2} e^{-\frac{s^2}{2}} \frac{\sqrt{\pi}}{2} = \sqrt{2\pi} e^{-\frac{s^2}{2}}$$

Properties of Fourier Transform:

(1) <u>Linear Property</u>: If f(x) and g(x) are two functions having Fourier Transforms then

)

where and are constants.

Proof:

=

=

(2) Change of Scale Property: If the Fourier Transforms of f(x) , then the Fourier Transforms of f(ax)

Proof: Since therefore F(s)= …….(1)

Now [Putting t=ax i.e. dt=adx]

=

== by (1)

(3) Shifting Property: If the Fourier Transforms of f(x), then the Fourier Transforms of f(x-a),

Proof:

= [put x-a=t]

=

==.

(4) Modulation Property: If the Fourier Transforms of f(x) , then the Fourier Transforms of

Proof:

=

=

==

Illustration: Let we are to evaluate the Fourier Transforms of the function

f(x)=4

=4()-5 () ………(1)

Now, by formula , ()=

By shifting property, () = ()

= ()

= ()

= , by change of scale property

Where F(s)=(

=

()=

So , from (1) we get ,

=

Lecture 10.

(5) Fourier Transform of Derivatives: If f(x), all tend to 0 as and dx converges for all j then

On multiplication by x.

If then -i

Proof:

Now

=

=i

=i

Example: Find the Fourier transform of

Solution: Now =

Therefore ,

(by change of scale property)

=

…………..(1)

Or,

Or,(-8x)=-is

Or, -8 (x)=-is. by (1)

Or,

(6) **Convolution Theorem:** If f(x) and g(x) are two functions defined on the interval (-) then f*g= is called the convolution of the two functions f and g.

The Fourier Transform of the convolution of f(x) and g(x) is the product of their Fourier Transform

i.e. (f*g)=

Example: Find the function f(x) from the following integral equation

f(x)=g(x)+

Solution: Taking Fourier Transform on both side we get

=

= , by Convolution theorem

Or, F(s)=G(s)+F(s)H(s)

Or, F(s)=

So by Fourier integral theorem

where f(x) is continuous.

at the point of continuity.

Lecture 11.

verse Fourier transform or Fourier Integral:

In this article we discuss how a function can be found if its Fourier Transform is known. This is due to the following theorem.

Theorem 1: (Fourier integral theorem)

If a function f(x)

(i)satisfies Dirichlet's condition in every finite interval [-T,T]

(ii) exists finitely

Then where f(x) is continuous

= when f(x) is discontinuous at x

Where F(s) is the Fourier Transform of f(x).

Definition: The integral is called Inverse Fourier Transform or Fourier Integral of the function f(x) , where F(s) is the Fourier Transform of f(x)

So, at the point of continuity.

Fourier Sine and Cosine Integral Theorem:

Theorem: If a function (i) satisfies Dirichlet's condition in every finite interval [0, T]

(ii) (ii) exists finitely

Then

(a) where f(x) is continuous

= where f(x) is discontinuous

Where is the Fourier Sine Transform of f(x).

(b) where f(x) is continuous.

= where f(x) is discontinuous

Where is the Fourier Cosine Transform of f(x).

Fourier Sine Integral: The integral is called Inverse Fourier Transform or Fourier Sine Inverse of f(x) , where is Fourier Sine Transform of f(x).

Fourier Cosine Integral: integral is called Inverse Fourier Transform or Fourier Cosine Inverse of f(x) , where is Fourier Cosine Transform of f(x).

Inverse Property of Inverse Fourier Transform: If F(s) and G(s) are Fourier transform of th function f(x) and g(x) respectively then

Where are constants.

Examples:

1. Find the Fourier transform of the function

$f(x)=1-x^2$,

=0 ,

Hence evaluate

The Fourier Transform,

F(s)=

=

=

=2 [

= 2{[

=

==

Since f(s) is an even function therefore F(s)=2

By Fourier integral theorem,

Or,

Or, $=1-x^2$,

=0 ,

Putting x= we get

=1-=

Or,

i.e. , , .

Lecture 12:

2. Find the Fourier Cosine transform of the function

f(x)=cosx , 0<x<a

=0 ,x>a

Solution: The required transform,

=

=

=

3. Find the Fourier Sine transform of the function f(x)=

Solution: The required transform,

=

= putting sx=t i.e. , dx=

=

4. Find the function whose cosine transform is

Solution: Let f(x) be the required function. Then corresponding to f(x).

By Fourier cosine integral theorem,

where f(x) is continuous

Or, =

=

Or, f(x)= if x<a

= if x>a since,

EXERCISES

1. Find the Fourier Transform of

$f(x)=x^2$,

=0 ,

2. If the Fourier transform, F(s)= evaluate f(x)
3. Find Fourier Cosine transform of f(x)= and hence derive fourier sine transform of the function
4. Evaluate
5. Evaluate the integral
6. Prove that =

   =0 , x>
7. Find Fourier sine and Cosine transform of and using the inversion formula recover the original function, in both the cases.
8. Verify the Convolution theorem for the function

   f(x)=1 ,

   =0 ,

   And g(x)= 1 ,

   =0 ,
9. Find the Fourier Integral of the function

   f(t)=1

   =0 ,
10. Evaluate

MODULE-II

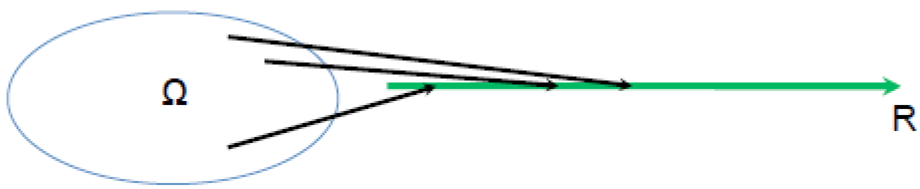
(Probability Distributions)

LECTURE 13:

Basic concepts of Random Variable and its Probability Distribution

11.1. Random variable

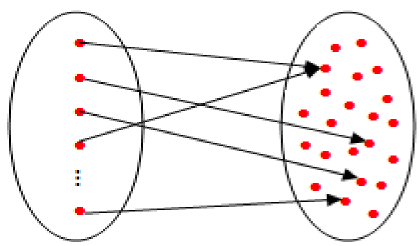
We are often interested not in the exact outcome of a random experiment but only some consequence of it (for example I toss 3 coins but I only care about how many heads occur not the exact outcome). In informal way of thinking, a random variable is an assignment of a value (number) – real or complex - to every outcome in the sample space. This number is called the numerical value or the experimental outcome of the random variable. But, we will restrict our discussion only on real valued random variable, i.e., the random variable which is a function from the sample space to the set of real numbers.



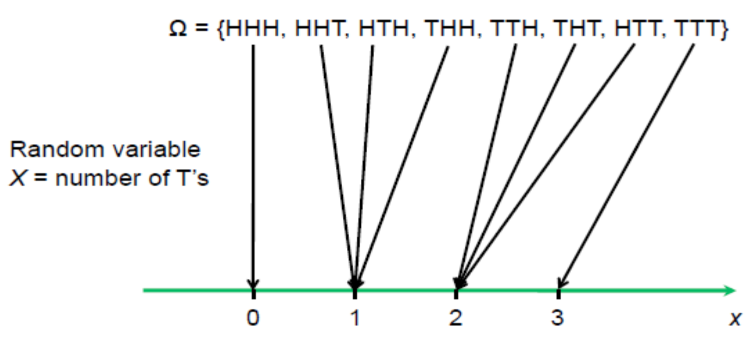
Definition 11.1. (*Random Variable*)

A random variable is a real-valued function defined on the sample space of a random experiment, i.e., a function from to .

Notation: Generally, the R.V.s are denoted by uppercase letters of English alphabets, viz., *X, Y, Z, etc*. and their corresponding values are denoted by *x, y, z*, etc., respectively. Thus, if X is a R.V., then . So, , .



Example 11.1: Let, a coin is tossed three times and *X* is the number of T's in the outcome.



Remarks:

- An (informal) way of thinking is to regard a random variable as a measure about the outcome which takes a real number (e.g. how many heads occur in a coin tossing experiment) or a measurement made on the outcome.
- For a given a random variable on , statements like “” or “” are events. Specifically, “” is the event , i.e. the set of all outcomes in for which   takes the value 3. Similarly, “” is the event i.e. that is the set of all outcomes in for which takes a value at most 3.
- For a random variable , the range of is the set of all values taken by . We denote it by Range and is commonly known as *spectrum* of the random variable.
- Functions of random variables: Any function you are likely to run across of a random variable or random variables is a random variable. So if and are random variables, then , , and are all random variables.

11.2. Types of random variables

Depending on the nature of possible values of the random variable, random variables are categorized into following two type:

- Discrete random variables – random variables that take on either finite or countably infinite number of values. For example, Number of phone messages awaiting me.
- Continuous random variables – random variables that take on a continuously infinite number of values. For example: Weight of a people selected randomly

11.3. Distribution of a random variable

The distribution of a random variable describes the probability that it takes on various values. In informal way of speaking, the probability distribution of a random variable is a representation which tells us which are the possible values and how the total probability is distributed over the several possible values.

Based on type of random variable, probability distribution of random variables are categorized as *discrete probability distribution* & *continuous probability distribution*.

Worked out Problem:

Problem 11.1. Find the spectrum for each of the following random variables.

(i) I toss a coin 100100 times. Let X be the number of heads I observe.

(ii) I toss a coin until the first heads appears. Let YY be the total number of coin tosses.

(iii) The random variable T is defined as the time (in hours) from now until the next earthquake occurs in a certain city.

Solution:

(i) The random variable can take any integer from 0 to 100, so .

(ii) The random variable Y can take any positive integer, so .

(iii) The random variable T can in theory get any positive real number, so .

Problem 11.2. Find the probability distribution of the number of tosses required, when an unbiased coin is tossed till a head appears for the first time

Solution: Let, denotes the number of of tosses required, when an unbiased coin is tossed till a head appears for the first time.

The set of possible values of are .

Now,

and so on.

The required probability distribution is as follows

LECTURE 14.

Discrete Probability Distribution

12.1. Introduction

A random variable can be either discrete or continuous; and accordingly a probability distribution can be discrete or continuous. In this lecture, we cover the first type, i.e., discrete probability distributions.

12.2. Probability Mass Function

A discrete probability distribution is characterized by it's *probability mass function* (p.m.f).

Definition 12.1. (*Probability Mass Function*)

The probability mass function of a discrete random variable is a function , defined as follows:

Properties of p.m.f.

If is the p.m.f. of a discrete random variable , then satisfies the following properties:

(i)

(ii)

12.3. Cumulative Distribution Function

The cumulative distribution function of a random variable can be considered as an alternative method to describe the distribution of random variables. The advantage of cumulative distribution function the is that it can be defined for any kind of random variable (discrete, continuous, and mixed).

Definition 12.2. (*Cumulative Distribution Function*)

The *cumulative distribution function* (c.d.f.) or simply *Distribution Function* (d.f) of a discrete random variable is a function , defined as follows:

where is the p.m.f. of the distribution.

Properties of c.d.f.

satisfies the following properties:

(i)

(ii) and

(iii)

LECTURE 15.

Expectation, Variance & S.D. of Discrete Random Variables

Definition 13.1. (*Expectation*)

Let be a discrete random variable and be the probability mass function. Then expected value of , denoted by , is defined by

provided this sum converges absolutely. We often refer to the expected value as the mean, and denote by for short. If the above sum does not converge absolutely, then we say that does not have an expected value.

Worked out Problem:

Problem 13.1. Let an experiment consist of tossing a fair coin three times. Find the expected number of heads.

Solution: Let denote the number of heads which appear. Then the possible values of are 0, 1, 2 and 3. The corresponding probabilities are 1/8, 3/8, 3/8, and 1/8. Thus, the expected value of is as follows:

Definition 13.2. (*Variance*)

Let be a discrete random variable and be the probability mass function. Then variance of , denoted by , is defined by

provided this sum converges absolutely.

Worked out Problem:

Problem 13.2. Let an experiment consist of tossing a fair coin three times. Find the variance of the number of heads appears.

Solution: Let denote the number of heads which appear. Then the possible values of are 0, 1, 2 and 3. The corresponding probabilities are 1/8, 3/8, 3/8, and 1/8. Thus, the expected value of is as follows:

Definition 13.3. (*Standard Deviation*)

Standard deviation of a discrete random variable is the positive square root of variance, i.e., .

LECTURE 16.

Continuous Probability Distribution

14.1. Introduction

Probability distribution which correspond a continuous random variable is known as continuous probability distribution. In this lecture, we cover the continuous probability distribution.

14.2. Probability Density Function

A continuous probability distribution is characterized by it's *probability density function* (p.d.f.).

Definition 14.1. (*Probability Density Function***)**

The probability density function of a continuous random variable is a function , defined as follows:

.

<u>Note:</u>

- does not give the probability that the continuous random variable takes on the value .
- is approximately equal to the probability that takes on a value in an interval of length about .
- For a continuous random variable ,

<u>Properties of p.d.f.</u>

If is the p.d.f. of a continuous random variable , then satisfies the following properties:

(i)

(ii)

14.3. Cumulative Distribution Function

The cumulative distribution function for continuous random variables is just a straightforward extension of that of the discrete case. All we need to do is replace the summation with an integral.

Definition 14.2. (*Cumulative Distribution Function*)

The *cumulative distribution function* (c.d.f.) or simply *Distribution Function* (d.f) of a continuous random variable is a function , defined as follows:

where is the p.d.f. of the distribution.

Properties of c.d.f.

satisfies the following properties:

(i)

(ii) and

(iii)

Worked out Problem:

Problem 14.1.

A probability density function is given by:

$f(x) = 0$ for $x < 1$

$f(x) = b / x^2$ for $1 < x < 5$

$f(x) = 0$ for $x > 5$

a) What is the value of b?

b) From this obtain the probability that X is between 2 and 4.

c) What is the probability that X is exactly 2?

d) Find the cumulative distribution function of X.

Solution:

We know that,

$$\int_{-\infty} 0\,dx + \int_1^5 \frac{b}{x^2}\,dx + \int_5^{\infty} 0\,dx = 1$$

$$\text{Therefore } \int_1^5 b\,x^{-2}\,dx = 1$$

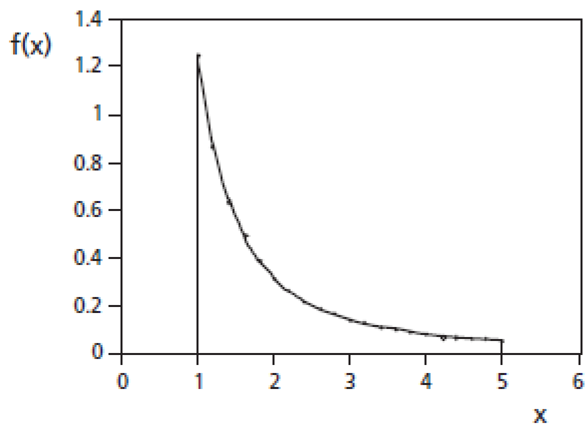
$$\left[-b\,x^{-1}\right]_1^5 = 1$$

$$-b\left(\frac{1}{5} - 1\right) = 1$$

$$\frac{4}{5}b = 1$$

$$b = 1.25$$

Then a graph of the density function for this example is shown below:



b) $\Pr[2 < X < 4] = \int_2^4 1.25\, x^{-2} dx$

$$= \left[-1.25\, x^{-1}\right]_2^4$$

$$= (-1.25)\left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= 0.3125$$

c) $\Pr[X = 2 \text{ exactly}] = \int_2^2 1.25\, x^{-2}\, dx$

$$= \left[-1.25\, x^{-1}\right]_2^2$$

$$= 0$$

d) For $x_1 < 1$: $F(x_1) = \int_{-\infty}^{x_1} 0\, dx = 0$

For $1 < x_1 < 5$: $F(x_1) = 0 + \int_{1}^{x_1} 1.25\, x^{-2}\, dx$

$$= (-1.25)\left[x^{-1}\right]_1^{x_1}$$

$$= (-1.25)\left(\frac{1}{x_1} - 1\right)$$

$$= 1.25\left(1 - \frac{1}{x_1}\right)$$

For $5 < x_1 < \infty$: $F(x_1) = \int_{-\infty}^{x_1} f(x)dx$

$$= \int_0^1 0\, dx + \int_1^5 1.25x^{-2}dx + \int_3^{x_1} 0\, dx$$

$$= 0 + (-1.25)\left[x^{-1}\right]_1^5 + 0$$

$$= (-1.25)\left(\frac{1}{5} - 1\right)$$

$$= 1$$

Then to summarize, the cumulative distribution function of X is:

	0	for $x_1 < 1$
	$1.25\left(1 - \frac{1}{x_1}\right)$	for $0 < x_1 < 5$
and	1	for $x_1 > 5$

LECTURE 17.

Expectation, Variance & S.D. of Continuous Random Variables

Definition 15.1. (*Expectation*)

Let be a continuous random variable and be the probability density function. Then expected value of , denoted by , is defined by

Definition 15.2. (*Variance*)

Let be a continuous random variable and be the probability density function. Then variance of , denoted by , is defined by

provided this sum converges absolutely.

Definition 15.3. (*Standard Deviation*)

Standard deviation of a continuous random variable is the positive square root of variance, i.e., .

LECTURE 18.

Binomial Distribution

Definition: A random variable is said to follow **binomial distribution**, if it can assume only finite number of non-negative integral values and it's probability mass function (p.m.f.) is given by

where the two independent constants and in the distribution are known as the parameters of the distribution.

Illustration: Let a random experiment be performed repeatedly, each repetition being called a trail and let the occurrence of an event in a trail be called a 'success' and it's non-occurrence be 'failure'. Now, in a series of independent trails, if the probability of 'success' in each trail is a constant and the probability of 'failure' in each trail is (where,), then the probability of successes (and obviously failures) is given by the binomial distribution. So, it is clear that i)

Physical Condition for Binomial Distribution: We get the binomial distribution under the following experimental conditions:

i. Each trail results in two exhaustive and mutually disjoint outcomes, termed as success and failure.
ii. The number of trails is finite.
iii. The trails are independent of each other. Thus, the probability of success is same in each trail.

NOTE:

1. Binomial distribution is a discrete distribution as can assume only finite number of isolated values.
2. Any random variable which follows binomial distribution is known as binomial variate and is denoted by , where and are the parameters of the distribution. **is the mean, is the variance** and **is the standard deviation (s.d.)** of the normal distribution.
3. Let a random experiment be performed repeatedly, each repetition being called a trail and let the occurrence of an event in a trail be called a 'success' and it's non-occurrence be 'failure'. Now, in a series of independent trails, if the probability of 'success' in each trail is a constant and the probability of 'failure' in each trail is (where,), then the probability of successes (and obviously failures) is given by the binomial distribution.

4. **Physical Condition For Binomial Distribution:** We get the binomial distribution under the following experimental conditions:
    i. Each trail results in two exhaustive and mutually disjoint outcomes, termed as success and failure.
    ii. The number of trails is finite.
    iii. The trails are independent of each other. Thus, the probability of success is same in each trail.

Worked out Problem:

Problem 16.1. A biased coin is tossed 6 times. The probability of heads on any toss is 0.3. Let X denote the number of heads that come up. Calculate: (i) $P(X = 2)$ (ii) $P(X = 3)$ (iii) $P(1 < X \leq 5)$.

Solution: If we call heads a success then this X has a binomial distribution with parameters n = 6 and p = 0.3.

(i) .

(ii)

(iii)

Problem 16.2. A quality control engineer is in charge of testing whether or not 90% of the DVD players produced by his company conform to specifications. To do this, the engineer randomly selects a batch of 12 DVD players from each day's production. The day's production is acceptable provided no more than 1 DVD player fails to meet specifications. Otherwise, the entire day's production has to be tested.

(i) What is the probability that the engineer incorrectly passes a day's production as acceptable if only 80% of the day's DVD players actually conform to specifications?

(ii) What is the probability that the engineer unnecessarily requires the entire day's production to be tested if in fact 90% of the DVD players conform to specifications?

Solution: Let X denote the number of DVD players in the sample that fail to meet specifications. X has a binomial distribution with parameters and .

(i) X has a binomial distribution with parameters and .

(ii) X has a binomial distribution with parameters and .

LECTURE 19.

Poisson Distribution

Definition: A random variable is said to follow **Poisson distribution**, if it can assume only infinite number of non-negative integral values and it's probability mass function (p.m.f.) is given by

where the independent constant in the distribution are known as the parameters of the distribution.

NOTE:

1. Poisson distribution is a discrete distribution as can assume infinite number of isolated values.
2. Any random variable which follows Poisson distribution is known as Poisson variate and is denoted by , where is the parameters of the distribution.
3. Poisson distribution is such a discrete distribution where
4. **Physical Condition for Poisson Distribution:** Poisson distribution occurs when there are events which are do not occur as outcomes of definite number of trails (unlike that binomial distribution) of an experiment but which occur at random points of time and space. For examples,
   (i) the number of hits to a web site in a day;
   (ii) the number of calls that arrive in any one day on your mobile phone;
   (iii)the number of jobs arriving in any one minute in a busy computer centre;
   (iv)the number of messages arriving to a computer server in any one hour.

<u>Worked out Problem:</u>

Problem 17.1. The number of calls coming per minute into a hotels reservation center is Poisson random variable with mean 3. Find the probability that no calls come in a given 1 minute period.

Solution:

(a) Let X denote the number of calls coming in that given 1 minute period.

Thus,

Problem 17.2. Consider a computer system with Poisson job-arrival stream at an average of 2 per minute. Determine the probability that in any one-minute interval there will be

(i) 0 jobs;

(ii) exactly 2 jobs;

(iii)at most 3 arrivals.

Solution:

Let X denote the number of jobs -arrival in any one-minute.

(i) No job arrivals:

(ii) Exactly 3 job arrivals:

(iii)At most 3 arrivals:

LECTURE 20.

Normal and Standard Normal Distribution

Definition: A random variable is said to follow **normal distribution**, if it can assume any real real number and it's probability density function (p.d.f.) is given by

where the two independent constants and in the distribution are known as the parameters of the distribution.

NOTE:

1. Normal distribution is a continuous distribution as can assume any value within the interval .
2. Any random variable which follows normal distribution is known as normal variate and is denoted by , where and are the parameters of the distribution. **is the mean** and **is the standard deviation (s.d.)** of the normal distribution.
3. **Normal Curve:** The graph corresponding to the p.d.f. of a normal distribution with parameters and is known as normal curve. The normal curve is a bell-shaped curve which is symmetric about the ordinate . The top of the bell is directly above . For large values of , the curve tends to flatten out and for small values of , it has a sharp peak.
4. **Distribution Function:** The cumulative distribution function (c.d.f.) or distribution function (d.f.) of a normal distribution with parameters and , is denoted by , is defined as follows:

<u>Standard Normal Distribution</u>

Definition: A normal distribution with parameters 0 and 1 is called **standard normal distribution**.

NOTE:

1. If be a continuous random variable which follows standard normal distribution, then is known as standard normal variate and is denoted by . Thus, the mean and standard deviation of a standard normal distribution are respectively 0 and 1.
2. Generally, the p.d.f. of any continuous probability distribution is denoted by , but the p.d.f. of a standard normal distribution is always denoted by and is defined as follows (because here and)
3. **Standard Normal Curve:** The graph corresponding to the p.d.f. of a standard normal distribution known as standard normal curve. The standard normal curve is a bell-shaped curve which is symmetric about the ordinate (because here). The top of the bell is directly above .
4. .

Since, standard normal curve is symmetric about the ordinate ,

5. Generally, the c.d.f. of any continuous probability distribution is denoted by , but the c.d.f. of a standard normal distribution is always denoted by and is defined as follows (because here and)

   Thus, for ,

LECTURE 21.

Transformation of a Normal Distribution into Standard Normal Distribution

Problem related to normal distribution is solved by transforming the problem into a problem of standard normal distribution.

Theorem: If and , then . Thus, the p.d.f. corresponding to is as follows

and the c.d.f. corresponding to is as follows

Technique to solve problems related to Normal Distribution

Suppose and we have to find the following probabilities

I

II

III

Let, . Then .

Therefore, the p.d.f. corresponding to is as follows

I

II

III

IV

Worked out Problem:

Problem 19.1. If , then find the followings:

(i)

(ii)

Solution: Here, .

(i)

.

(ii)

.

Problem 19.2. Suppose the diameter of a certain car component follows the normal distribution with . Find, if we randomly select one of these components, the probability that its diameter will be larger than 13.4 mm.

Solution:

Here, .

.

LECTURE 22.

Binomial Approximation to Poisson and Normal Distribution

20.1. Binomial Approximation To Poisson Distribution

When the value of in a binomial distribution is large and the value of is very small, the binomial distribution can be approximated by a Poisson distribution. The larger the and the smaller the , the better is the approximation.

NOTE:

1. The Poisson random variable theoretically ranges from to . However, when used as an approximation to the binomial distribution, the Poisson random variable - the number of successes out of observations - cannot be greater than the sample size .
2. The binomial distribution tends towards the Poisson distribution when , and stays constant.

Worked out Problem:

Problem 20.1.

5% of the tools produced by a certain process are defective. Find the probability that in a sample of 40 tools chosen at random, exactly three will be defective. Calculate a) using the binomial distribution, and b) using the Poisson distribution as an approximation.

Answer: a) For the binomial distribution with $n = 40$, $p = 0.05$,

$$\Pr[R = 3] = {}_{40}C_3\,(0.05)^3(0.95)^{37}$$
$$= \frac{(40)(39)(38)}{(3)(2)(1)}(0.05)^3(0.95)^{37}$$
$$= 0.185$$

b) For the Poisson distribution, $\mu = (n)(p) = (40)(0.05) = 2.00$.

$$\Pr[R = 3] = \frac{(2.00)^3 e^{-2.00}}{(3)(2)(1)} = 0.180$$

20.2. Binomial Approximation To Normal Distribution

Worked out Problem:

Problem 20.2.

Stanford accepts 2480 students and each student has a 68% chance of attending. Let X = # students who will attend. $X \sim Bin(2480, 0.68)$. What is $P(X > 1745)$?

Solution:

$E[X] = np = 1686.4$. $Var(X) = np(1-p) = 539.7$. $\sigma = \sqrt{(Var(X))} = 23.23$. We can thus use a Normal approximation: $Y \sim \mathcal{N}(1686.4, 539.7)$.

$$P(X > 1745) \approx P(Y > 1745.5) = P\left(\frac{Y - 1686.4}{23.23} > \frac{1745.5 - 1686.4}{23.23}\right) = 1 - \Phi(2.54) = 0.0055$$

Some Problems (MCQ)

1. Four coins are tossed. Expectation of number of heads is

(a) 1 (b)2
(c) 3 (d)4

2. A card is drawn at random from a well shuffled pack of 52 cards. The probability of getting a heart or a diamond is
   (a) 1 **(b)1/2**
   (c) 1/26 (d)3/13

3. Let A, B be two events and P($\bar{A}$) = 0.3, P(B) =0.4, P(A $\bar{B}$) = 0.5, then P(A+ $\bar{B}$) =
   (a) 0.5 **(b)0.8**
   (c) 1 (d) None of these

4. The probability that a leap year, selected at random, will contain 53 Sunday is
   (a) 1/7 (b)2/365
   (c) 2/7 (d) None of these

5. If A and B are two mutually exclusive events, then P(A+B) =
   (a) P(A)+P(B) (b) P(A) P(B)
   (c) P(A) –P(B) (d) None of these

6. If P(A) = 1/2, P(B) = 1/3, P(AB) = ¼, then the value of P(A$\cup$B) is
   (a) 6/7 (b) 3/7
   (c) 1 **(d) 7/12**

7. " Two mutually exclusive events are always independent ". This statement is
   (a) true **(b) false**

8. If $\bar{A}$ is the complementary event of A, then
   (a) P($\bar{A}$)=1-P(A) (b) P($\bar{A}$)=P(A)
   (c) P($\bar{A}$)=1+P(A) (d) None of these

9. An unbiased die is thrown. The probability that either an even number or a number greater than 2 will turn up is
   (a) 1/6 (b) 2/3
   (c) 5/6 (d) None of these

10. A man draw at random three balls from a bag containing 6 red and 5 green balls. The probability of getting the balls all red is
    (a) 6/11 (b) 3/22
    (c) 4/33 (d) 1/6

MODULE III

Calculus of Complex Variable

Lecture 23.

COMPLEX POINT

A collection of points in the complex plane (Argand plane) is called a point set . So the complex point set is nothing but the set of some complex numbers. For example the set { 1+2i , -1+6i , 0+3i } is a complex point set.

Neighbourhood of a point.

Let z_o be a point of the complex plane. The set of all points z satisfying the inequality $|z - z_0| < \in$ is called the neighbourhood of z_o and is denoted by $N(z_o, \in)$. For example $N(1+2i, 0.3)$ is a neighbourhood of the region inside the circle with centre (1,2) and Radius 0.3 excluding the points on the circumference.

If from the neighbourhood of a point z_o we exclude the point z_o itself then such a neighbourh hood is called the deleted neighbourhood of z_o. and is represented by $0 < |z - z_0| < \in$ and is denoted by $N^o(z_o, \in)$.

Limit Point.

A point z_o is called a limit point of a set S if every neighbourhood$N(z_o)$ contain at least one point of S other than z_o.

FuntionOf A Complex Variable

When a symbol z takes any one of the values of a set of complex numbers then z is called a complex variable.

Let D and R be any two non empty point sets in the complex plane. A complex variable w $\in$ R is said to be the function of a complex variable z $\in$ D ,if to every value of z corresponds oneor valus of w. Thus if w is a function of z , it is written as w = f(z).

If z = x+iy and w = u + iv then u and v are both functions of real variable and we may Write w = f(z) = u(x,y) + i v(x,y) . Here D is called Domain and R is called range of f.

Illustration 1. Consider the function f : S → C, given by f(z) = z^2 and where

S = {z C : |z| < 2} is the open disc with radius 2 and centre 0.

Using polar coordinates, it is easy to see that the range of the function is the open disc f(S) = {w C : |w| < 4} with radius 4 and centre 0.

Limits and Continuity.

The concept of a limit in complex analysis is exactly the same as in real analysis.

So, for example, we say that f(z) → L as z → z_o, or

$$\lim_{z \to z_o} f(z) = L$$

if, given any $\in > 0$, there exists δ > 0 such that |f(z) − L| < ∈ whenever 0 < |z – z_o| < δ.

Similarly, we say that a function f(z) is continuous at z_o if f(z) → f(z_o) as z → z_o. A similar qualification on z applies if z_o is a boundary point of the region S of definition of the function. We also say that a function is continuous in a region if it is continuous at every point of the region.

Note that for a function to be continuous in a region, it is enough to have continuity at every point of the region. Hence the choice of δ may depend on a point z_o in question. If δ can be chosen independently of z_o, then we have some uniformity as well.

Theorems on Continuity.

Theorem 1. A necessary and sufficient condition of a function f(z) = u + iv to be continuous At $z_0 = x_0 + iy_0$ is that u(x,y) and v(x,y) be continuous at (x_0 , y_0).

Theorem 2. If the function f(z) and g(z) are defined in D and continuous at z = z_0, then

(i) pf(z) +qg(z) (p , q are constants)
(ii) f(z)g(z)

(iii) $f(z)/g(z)$ if $g(z) \neq 0$

are also continuous at $z = z_0$.

Example 1.Prove that $\lim_{z\to 0}\frac{\bar{z}}{z}$ does not exist.

Solution.Let $z = x+iy$ then $\bar{z}$ = x-iy therefore $\frac{\bar{z}}{z} = \frac{x-iy}{x+iy}$.

Let $z \to 0$ along x axis . Then y=0 thus $\lim_{z\to 0}\frac{\bar{z}}{z} = \lim_{x\to 0}\frac{x}{x} = 1$.

Next let $z \to 0$ along y axis then x=0

Therefore $\lim_{z\to 0}\frac{\bar{z}}{z} = \lim_{y\to 0}\frac{-iy}{y} = -1$.

Thus $\lim_{z\to 0}\frac{\bar{z}}{z}$ has different values along different path.

Hence $\lim_{z\to 0}\frac{\bar{z}}{z}$ does not exist.

Example 2. Test the continuity at origin of the following function

$$f(z) = \frac{xy^3}{x^2+y^6} \quad \text{for } z \neq 0$$
$$= 0 \quad \text{for } z = 0$$

Solution. Let $f(z) = u(x,y) + iv(x,y)$, Then

$$u(x,y) = \frac{xy^3}{x^2+y^6}, \quad (x,y) \neq (0,0)$$
$$= 0 \quad (x,y) = (0,0)$$

$v(x,y) = 0 \quad \forall\ (x,y)$

Let $(x,y) \to (0,0)$ along the curve $x=my^3$.

Therefore, $$\lim_{(x,y)\to(0,0)} u(x,y) = \lim_{y\to 0}\frac{my^3y^3}{m^2y^6+y^6} = \frac{m}{1+m^2}$$

Thus, $\lim_{(x,y)\to(0,0)} u(x,y)$ has different values for different values of m

Hence , $\lim_{(x,y)\to(0,0)} u(x,y)$ does not exist . Therefore u(x,y) is not continuous at (0,0). Consequently f(z) is not continuous at z= 0.

Lecture 24.

Differentiability

A function w = f(z) defined in a certain domain D is said to be differentiable at $z = z_o$ if the

Limit $\lim_{z\to z_o}\dfrac{f(z)-f(z_o)}{z-z_o}$ exists

Theorem 1. If f(z) = u(x,y) + iv(x,y) is differentiable then u and v are also differentiable .

Moreover, $f^{/}(z) = u_x+iv_x$ and $f^{/}(z) = -iu_x + v_y$.

Proof. $f^{/}(z) = \lim_{\Delta Z\to 0}\dfrac{f(z+\Delta z)-f(z)}{\Delta z}$.

Here $\Delta z \to 0$ along any path . Let $\Delta z = h \to 0$ along real axis . Then

$$f^{/}(z) = \lim_{\Delta Z\to 0}\frac{f(z+h)-f(z)}{h}$$

$$= \lim_{h\to 0}\frac{\{u(x+h,y)+iv(x+h,y)\}-\{u(x,y)+iv(x,y)\}}{h}$$

$$= \lim_{h\to 0}\left[\frac{\{u(x+h,y)-u(x,y)\}}{h}+i\frac{v(x+h,y)-v(x,y)}{h}\right] = u_x + iv_x$$

Next let $\Delta z \to 0$ along imaginary axis i.e, $\Delta z = ik$ and $k \to 0$

Then, $f^{/}(z) = \lim_{k\to 0}\dfrac{f(z+ik)-f(z)}{ik}$

$$= \frac{\lim_{k\to 0}\{u(x,y+k)+iv(x,y+k)\}-\{u(x,y)+iv(x,y)\}}{ik}$$

$$= \frac{\lim_{k\to 0}\{u(x,y+k)-u(x,y)\}}{ik} + i\frac{\lim_{k\to 0} v(x,y+k)-v(x,y)}{ik}$$

$$= -I\frac{\lim_{k\to 0}\{u(x,y+k)-u(x,y)\}}{k} + \frac{\lim_{k\to 0} v(x,y+k)-v(x,y)}{k} = -iu_y+v_y$$

Theorem 2. If a function is differentiable at a point , then it is continuous at that point.

Proof. Left as an exercise

<u>Analytic Functions</u>

If a function f(z) be such that $f^{/}(z)$ exists at every point of the domain D then f(z) is said to be analytic in D

Lecture 25.

<u>Cauchy Riemann conditions</u>

The necessary conditions for w = f(z) = u(x,y) + iv(x,y) is analytic at any point z = x+iy

Of its domain D is that the four partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial y}$ should exist and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Example3. For the function defined by f(z) = $\sqrt{|xy|}$ show that the Cauchy Riemann Equation are satisfied at (0,0) but the function is not differentiable and analytic at that Point.

<u>Solution.</u> Let f(z) = u(x,y) + iv(x,y).

Then u(x,y) = $\sqrt{|xy|}$ and v(x,y) = 0

Now at the origin,

$$\frac{\partial u}{\partial x} = \lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0}\frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y} = \lim_{x\to 0}\frac{0-0}{y} = 0$$

Similarly $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 0$

Hence Cauchy Riemann equation is satisfied at origin.

Again $f'(0) = \lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{(x,y)\to 0}\frac{\sqrt{|xy|}}{x+iy}$.

Let $z \to 0$ along the straight line y=mx. Then

$f'(0) = \lim_{x\to 0}\frac{\sqrt{|mx^2|}}{x+imx} = \frac{\sqrt{|m|}}{1+im}$ which have different values for different m. Hence $f'(0)$ does not exist. Thus the function is not differentiable at the origin.

<u>Laplace's Equation</u>

A partial differential equation of the form $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ =0 is called Laplace's Equation

<u>Harmonic Function</u>

A function f(x,y) which possesses continuous partial derivatives of first and second orders And satisfies Laplace Equation is called Harmonic function.

<u>Conjugate Harmonic Function</u>

If the two harmonic functions u(x,y) and v(x,y) satisfy the Cauchy Riemann equations then they are called Conjugate harmonic functions

Lecture 26.

COMPLEX INTEGRATION

Curve represented by a complex variable

Let z = x+iy be a complex variable. Since a fixed z represents a point (x,y) in Argand Plane, so as z varies the point (x,y) moves on the plane and makes a locus or a curve. We say this curve, say C is represented by the complex variable z.

Parametric Representation

Let z = x+iy be a complex variable. If $x = \phi(t)$ and $y = \psi(t)$

Where t is a real variable then there is a relation between x and y . This relation gives a Curve , C (say)

Then we say $z = \phi(t) + i\psi(t)$ gives the curve C , t is a parameter.

Simple Curve.

A curve C is called simple if it does not intersect itself. So a curve C : $z = \phi(t) + i\psi(t)$

Is simple if $t_1 \neq t_2$ implies $z(t_1) \neq z(t_2)$.

Closed Curve.

A simple curve is called closed if the two end points of the curve coincide

Smooth Curve.

A curve C is called smooth if it possess unique tangent at every point.

Contour or Piecewise smooth curve.

A curve is called contour or piecewise smooth if it is comprised of a finite number of smooth curves.

Cauchy's Theorem.

Let f(z) be an analytic function and $f'(z)$ is continuous at each point within the domain D bounded by a closed contour C. Then,

$$\oint_c f(z)dz = 0$$

Proof. Let f(z) = u(x,y) + iv(x,y).

As f(z) is an analytic function , so by Cauchy Riemann condition,

We have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Also f′(z) is continuous and as $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$.

So u, v and their partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are all continuous within and on C.

So Greens theorem can be applied. $\oint_c f(z)dz = \oint_c (u + iv)(dx + idy)$

$$= \oint_c (udx - vdy) + i(vdx + udy) = \oint_c (udx - vdy) + i\oint_c (vdx + udy)$$

$$= \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy + i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dxdy$$, by Greens Theorem

$$= \iint_D \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y} \right) dxdy + i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x} \right) dxdy$$

= 0 +i.0

Lecture 27.

Cauchy Goursat Theorem.

Let f(z) be analytic function within and on a simple closed contour C. Then

$$\oint_C f(z)dz = 0$$

Formula 1. $\oint_C \frac{dz}{z-\alpha} = 0$ if C is any simple closed curve and $z = \alpha$ is an exterior point of C

As $z = \alpha$ is an exterior point of C so $f(z) = \frac{1}{z-\alpha}$ is an analytic function

Everywhere within and on C. Hence by Cauchy's theorem $\oint_C f(z)dz = 0$

i.e, $\oint_C \frac{dz}{z-\alpha} = 0$.

Formula 2. $\oint_C \frac{dz}{z-\alpha} = 2\pi i$, if C is any simple closed curve and $z = \alpha$ is an interior point of C

Formula 3. $\oint_C \frac{dz}{(z-\alpha)^n} = 0$, $n = 2,3,4,\ldots\ldots..$

Where α is an interior point of any simple closed curve C.

As in formula 1 we can write

$\oint_C \frac{dz}{(z-\alpha)^n} = \oint_{C_1} \frac{dz}{(z-\alpha)^n}$ where C_1 is a circle lying within C and the equation

Of the circle C_1 is $z - \alpha = \rho$

On the circle $|z-\alpha| = \rho$,

We have, $z - \alpha = \rho(\cos\theta + i\sin\theta) = \rho e^{i\theta}$

i.e, $z = \alpha + \rho e^{i\theta}$ where θ varies from 0 to 2π

Therefore $dz = \rho i e^{i\theta} d\theta$.

Thus $\oint_{C_1} \frac{dz}{(z-\alpha)^n} = \int_0^{2\pi} \frac{i\rho e^{i\theta}}{\rho^n e^{in\theta}} = \frac{i}{\rho^{n-1}} \left[\frac{e^{(1-n)i\theta}}{(1-n)} \right]_0^{2\pi} = 0.$

Hence, $\oint_C \frac{dz}{(z-\alpha)^n} = 0 \qquad n = 2,3,\ldots\ldots..$

Lecture 28.

Cauchy Integral Formula

Theorem 1. If f(z) is analytic within and on a simple closed curve C and α is any point within C. Then $f(\alpha) = \frac{1}{2\pi 1}\oint_C \frac{f(z)}{z-\alpha}dz$.

Proof. As $\frac{f(z)}{z-\alpha}$ is analytic everywhere within C except at $z=\alpha$, we draw a circle C_1 With centre at $z=\alpha$ and radius r so that C_1 lies wholly within C. Then $\frac{f(z)}{z-\alpha}$ is analytic Within the annular region bounded by C and C_1 .Hence ,

$\oint_C \frac{f(z)}{z-\alpha}dz = \oint_{C_1} \frac{f(z)}{z-\alpha}dz$. Now the equation of the circle C_1 is $|z-\alpha| = r$ thus $z-\alpha = re^{i\theta}$ where θ varies from 0 to 2π

Therefore $dz = rie^{i\theta}d\theta$

Thus $\oint_{C_1} \frac{f(z)}{z-\alpha}dz = \int_0^{2\pi} \frac{f(\alpha+re^{i\theta})}{re^{i\theta}} rie^{i\theta}d\theta$

Now taking limit as $r \to 0$ on both sides of above equation we have,

$$\oint_{C_1} \frac{f(z)}{z-\alpha}dz = i\lim_{r\to 0}\int_0^{2\pi} f(\alpha + re^{i\theta})d\theta = if(\alpha)\int_0^{2\pi} d\theta = 2\pi i f(\alpha)$$

Hence $f(\alpha) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\alpha}dz$.

NOTE : When α is an exterior point of C then $\frac{f(z)}{z-\alpha}$ is analytic within and on C.

So by Cauchy's theorem $\oint_C \frac{f(z)}{z-\alpha}dz = 0$

Lecture 29.

Cauchy Integral Formula

Theorem 2. If f(z) is analytic within and on a closed curve C , then the derivative of f(z) at an Interior point α of C is given by $f^{/}(\alpha) = \dfrac{1}{2\pi i}\oint_C \dfrac{f(z)}{(z-\alpha)^2}dz$.

Theorem 3. If f(z) is analytic within and on a closed curve C , then the nth order derivative of f(z) at any interior point α of C is given by

$$f^{n}(\alpha) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-\alpha)^{n+1}}dz.$$

ILLUSTRATIVE EXAMPLES

Example 1. Evaluate $\oint_C \dfrac{e^z}{(z+1)(z+2)}dz$ Where C is the circle $|z-1| = 4$.

Solution. Here f(z) = e^z is analytic within and on the circle $|z-1| = 4$ and z = -1, -2 are the interior Points of C. Now $\dfrac{1}{(z+1)(z+2)} = \dfrac{1}{z+1} - \dfrac{1}{z+2}$.

Therefore $\oint_C \dfrac{e^z}{(z+1)(z+2)}dz = \oint_C \dfrac{e^z}{(z+1)}dz - \oint_C \dfrac{e^z}{(z+2)}dz$

$= 2\pi i \times e^{-1} - 2\pi i \times e^{-2}$, by Cauchy's integral formula

$= 2\pi i(e^{-1} - e^{-2})$.

Example2. Evaluate $\oint_C \dfrac{\sin 3z + 2\cos z}{(z+\frac{\pi}{2})}dz$ if C is the circle $|z| = 5$

Solution. Here f(z) = sin3z +2cosz is analytic within and on the circle $|z| = 5$ and $z = \dfrac{-\pi}{2}$ lies Inside the circle C. Hence by Cauchy integral formula

$$\oint_C \frac{\sin 3z + 2\cos z}{(z+\frac{\pi}{2})}dz = 2\pi i(\sin 3(\frac{-\pi}{2}) + 2\cos(\frac{-\pi}{2})) = 2\pi i(1+2.0) = 2\pi i.$$

Example 3. Evaluate $\oint_C \dfrac{\cos^3 z}{(z-\frac{\pi}{4})^3}dz$ where C is the circle $|z| = 1$.

Solution. Let f(z) = $\cos^3 z$ which is analytic within and on C . Also $z = \dfrac{\pi}{4}$ lies inside the circle C .

Hence by Cauchy's integral formula $f^{''}(\alpha) = \dfrac{2!}{2\pi i}\oint_C \dfrac{f(z)}{(z-\alpha)^3}dz$.

We get

$$f''\left(\frac{\pi}{4}\right) = \frac{1}{\pi i} \oint_C \frac{\cos^3 z}{\left(z - \frac{\pi}{4}\right)^3} dz.$$

Now

$$\oint_C \frac{\cos^3 z}{\left(z - \frac{\pi}{4}\right)^3} dz = \pi i\left(6\cos\frac{\pi}{4}\sin^2\frac{\pi}{4} - 3\cos^3\frac{\pi}{4}\right)$$

$$= \frac{3\sqrt{2}\pi i}{4}.$$

Lecture 30.

Taylor's Theorem

Let f(z) be analytic at all points within a circle C_0 with centre z_0 and radius r_0. Then for every point z within C_0, we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots.. + \frac{f^n(z_0)}{n!}(z - z_0)^n + \ldots$$

$$= f(z_0) + \sum_{n=1}^{\infty} \frac{(z - z_0)^n}{n!} f^n(z_0).$$

If we put $z_0 = 0$ in the above series we get ,

$$f(z) = f(0) + \sum_{n=1}^{\infty} \frac{(z)^n}{n!} f^n(0)$$

, which is known as Maclaurin's series

Example 1. Find the Taylor's expansion of $f(z) = \frac{1}{(z+1)^2}$ about the point $z = -i$

Solution. To expand f(z) about z = -i in powers of z+i put z+i = t . Then

$$f(z) = \frac{1}{(t - i + 1)^2} = (1 - i)^{-2}\left[1 + \frac{t}{1 - i}\right]^{-2} = \frac{i}{2}\left[1 - \frac{2t}{1 - i} + \frac{3t^2}{(1 - i)^2} - \frac{4t^3}{(1 - i)^3} + \ldots\right]$$

$$= \frac{i}{2}\left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)(z+i)^n}{(1 - i)^n}\right]$$

Lecture 31.

Laurent's Series.

If a function f(z) is analytic in the annulus (ring shaped) region D bounded by two concentric circles C_1 and C_2 with the centre at the point $z = \alpha$ and radius r_1 and r_2. ($r_1 > r_2$), then for all z In D ,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n + \sum_{n=1}^{\infty} b_n (z-\alpha)^{-n}.$$

Where $a_n = \dfrac{1}{2\pi i}\oint_{C_1} \dfrac{f(z)}{(z-\alpha)^{n+1}}dz$, n= 0,1,2…

And $b_n = \dfrac{1}{2\pi i}\oint_{C_2} \dfrac{f(z)}{(z-\alpha)^{-n+1}}dz$, n= 1,2,…

Example 1. Expand the function f(z) = $\dfrac{z^2-1}{(z+2)(z+3)}$ when (i) $|z|<2$ (ii) $2<|z|<3$ and (iii) $|z|>3$

Solution. Let f(z) = $1+\dfrac{A}{z+2}+\dfrac{B}{z+3}$.

Then $z^2-1 = (z+2)(z+3)+A(z+3)+B(z+2)$.

Putting z = -2 , -3 we get A = 3, B =-8. Therefore f(z) = $1+\dfrac{3}{z+2}-\dfrac{8}{z+3}$.

When $|z|<2$,

$$f(z) = 1+\frac{3}{z+2}-\frac{8}{z+3} = 1+\frac{3}{2}\left(1+\frac{z}{2}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1+\frac{3}{2}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\ldots\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-\ldots\right)$$

$$= 1+\frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n-\frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n.$$

Example 2.Find the Laurent's series of the function $\dfrac{e^z}{(z-2)^3}$ about the point z=2.

Solution. Let z-2 = u then z = u +2.

$$\text{Therefore, } \frac{e^z}{(z-2)^3} = \frac{e^{u+2}}{(u)^3} = \frac{e^2}{(u)^3}\left(1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+\ldots\right)$$

$$= e^2\left(\frac{1}{u^3}+\frac{1}{u^2}+\frac{1}{2!}\frac{1}{u}+\frac{1}{3!}+.\frac{1}{4!}u+\ldots..\right)$$

$$= \frac{e^2}{(z-2)^3}+\frac{e^2}{(z-2)^2}+\frac{1}{2!}\frac{1}{z-2}+\frac{1}{3!}+\frac{1}{4!}(z-2)+\ldots$$

Lecture 32.

Zero and Singularities of an Analytic Function.

Definition. A point z = a is said to be a zero of an analytic function f(z) if f(a) = 0.
Order of zero. If f(z) is analytic in domain D and $a \in D$ then a is called zero of f(z) of

Order m if $f(a) = f'(a) = f''(a) = \ldots\ldots = f^{m-1}(a) = 0$ but $f^{m}(a) \neq 0$.

Thus from Taylors theorem $f(z) = \sum_{n=m}^{\infty} a_n (z-a)^n$ if a is mth order 0 of f(z) .

Singularities of an Analytic function

If a function f(z) is not analytic at the point z = a, then a is called the singularity or singular Point of f(z) .

Isolated and Non isolated singularity.

A singularity z = a of a function f(z) is said to be an isolated singularity if there is no other Singularity within a small neighbourhood of z = a.

If a singularity z = a of a function f(z) is not isolated then it is called non isolated singularity.

ILLUSTRATION

1. The function $f(z) = \frac{1}{z-1}$ is analytic everywhere except at z = 1. So z = 1 is the only Singularity of f(z). As the function f(z) has no other singularity, so z=1 is an isolated singularity.

2. The function $f(z) = \cot\frac{\pi}{z} = \frac{\cos\frac{\pi}{z}}{\sin\frac{\pi}{z}}$ is not analytic at z = 0 and at the points where $\sin\frac{\pi}{z} = 0$

   i.e. $\frac{\pi}{z} = n\pi$, n = 0,1,-1,2,-2,3,-3,…. I.e. $z = \frac{1}{n}$, n = 1,2,3,….

Essential Singularity and Pole

Principal Part of an Analytic Function.

Let α be an isolated singularity of an analytic function f(z) in a domain D. Now we draw a circle C_1 with center at $z = \alpha$ and radius as small as we please and another large concentric circle C_2 if any radius lying wholly within D. Then f(z) is analytic within the annular region between these two circles. Hence by Laurent's theorem, we have ,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n + \sum_{n=1}^{\infty} b_n (z-\alpha)^{-n}.$$

The second term in RHS is called the principal part of f(z) at the point $z = \alpha$.

Essential Singularities.

If the principal part of f(z) (at the isolated singularity α) contains an infinite no of terms, then The singularity $z = \alpha$ is called an essential singularity.

Pole

If the principal part of f(z) contains a finite number of terms say m, then the singularity $z = \alpha$ Is called a pole of order m. In this case the Laurent's series of f(z) is of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n + \frac{b_1}{(z-\alpha)} + \frac{b_2}{(z-\alpha)^2} + \ldots\ldots + \frac{b_m}{(z-\alpha)^m}, \quad b_m \neq 0$$

When m = 1 pole is said to be a simple pole.

ILLUSTRATION.

1. As $f(z) = e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \ldots$ contains an infinite number terms in negative power of z,

   So z = 0 is an essential singularity.

2. The function $f(z) = \frac{\sin(z-a)}{(z-a)^4} = \frac{1}{(z-a)^4}\left[(z-a) - \frac{1}{3!}(z-a)^3 + \frac{1}{5!}(z-a)^5 - \ldots\ldots\right]$

   $= \frac{1}{(z-a)^3} - \frac{1}{3!}(z-a) + \frac{1}{5!}(z-a) - \frac{1}{7!}(z-a)^3 + \ldots\ldots$ contains only two terms in negative power of z-a, so z = a is a pole of order 2.

Theorem 1. If an analytic function f(z) has a pole of order m at z = a then 1/f(z) has a Zero of order m at z = a.

Theorem 2. The limit point of the set of all poles of a function f(z) is a non isolated Essential singularity.

Theorem 3. The limit point of zeroes of a function f(z) is an isolated essential singularity.

ILLUSTRATION

Example 1. Find out the zeroes and discuss the nature of the singularities of

$F(z) = \frac{z-2}{z^2}\sin\frac{1}{z-1}$

Solution. Poles of f(z) are given by putting the denominator equal to zero i.e. $z^2 = 0$

So z = 0 is a pole of order 2.

Again the zeroes of f(z) are given by equating to zero the numerator of f(z)

i.e. $(z-2)\sin\frac{1}{z-1} = 0$ or $z - 2 = 0$ and $\sin\frac{1}{z-1} = 0$

Now z = 2 and $\sin\frac{1}{z-1} = 0$ gives $\frac{1}{z-1} = n\pi$ i.e. $z = 1 + \frac{1}{n\pi}$

Thus all zeroes of f(z) are given by $z = 2, 1 + \frac{1}{n\pi}$ where n = 1,-1,2,-2,3,-3…..

Also the limit point of zeroes given by $z = 1 + \frac{1}{n\pi}$ (n = 1,2,3,….) is z =1 . Hence z =1 is an isolated essential singularity.

Example 2. Find the Laurent series about the indicated singularity for the function

$F(z) = \dfrac{e^{z^2}}{z^3}$, $z = 0$

Solution $f(z) = \dfrac{e^{z^2}}{z^3} = \dfrac{1}{(z)^3}\left[1 + z^2 + \dfrac{(z^2)^2}{2!} + \dfrac{(z^2)^3}{3!}.....\right]$

$= \dfrac{1}{z^3} + \dfrac{1}{z} + \dfrac{z}{2!} + \dfrac{z^3}{3!}.......$

As the highest power of $\dfrac{1}{z}$ in the Laurent series is 3 , so z= 0 is a pole of order 3.

Lecture 33.

Residue Theorem

Let α be an isolated singularity of an analytic function f(z). Then by Laurents Theorem ,

We have $f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n + \sum_{n=1}^{\infty} b_n (z-\alpha)^{-n}$.

Where $a_n = \dfrac{1}{2\pi i}\oint_{C_1} \dfrac{f(z)}{(z-\alpha)^{n+1}} dz$, n= 0,1,2…

And $b_n = \dfrac{1}{2\pi i}\oint_{C_2} \dfrac{f(z)}{(z-\alpha)^{-n+1}} dz$, n= 1,2,…

The coefficient b_1 of $\dfrac{1}{(z-\alpha)}$ in the above infinite series is called the residue of f(z) at the Singularity $z = \alpha$ and is denoted by Res(α) .

ILLUSTRATION

Consider the function $f(z) = \dfrac{1}{(z+1)^2 (z-2)}$. Here $z = -1 , 2$ are the singularities of f(z)

Now $f(z) = \dfrac{1}{t^2 (t-1-2)}$ by putting $z + 1 = t$.

$= -\dfrac{1}{3t^2}\left(1-\dfrac{t}{3}\right)^{-1}$

$= -\dfrac{1}{3t^2}\left(1+\dfrac{t}{3}+\dfrac{t^2}{9}+...\right)$

$= -\dfrac{1}{3t^2} - \dfrac{1}{9}\dfrac{1}{t} - \dfrac{1}{27}$ -.....

$= -\dfrac{1}{3(z+1)^2} - \dfrac{1}{9}\dfrac{1}{(z+1)} - \dfrac{1}{27}$ -….

So the coefficient of $\dfrac{1}{(z+1)}$ is $-\dfrac{1}{9}$

Hence the residue at z = -1 is $-\frac{1}{9}$

Theorem. Let α be a pole of f(z) of order m . Then the residue of f(z) at z = α is given by

$$\frac{1}{(m-1)!}\lim_{z\to\infty}\frac{d^{m-1}}{dz^{m-1}}\left[(z-\alpha)^m f(z)\right]$$

Lecture 34.

Cauchy Residue Theorem

Let f(z) be analytic within and on a closed contour C except at a finite number of Singularities a_1 , a_2,…a_n and let R_1 , R_2,…R_n be respectively the residues of f(z) at these poles. Then , $\oint_c f(z)dz = 2\pi i(R_1 + R_2 + ...R_n)$.

Illustration

Ex 1. Use Cauchy Residue theorem to prove that

$$\oint_C \frac{z\cos z}{\left(z-\frac{\pi}{2}\right)^3}dz = -2\pi i$$

where C is the circle $|z-1| = 1$

Sol. Let f(z) = $\dfrac{z\cos z}{\left(z-\frac{\pi}{2}\right)^3}$

The Poles of f(z) is given by $\left(z-\frac{\pi}{2}\right)^3 = 0$ i.e. $z = \frac{\pi}{2}$ which is a pole of order 3 and lies within the circle $|z-1| = 1$. Hence by Cauchy's residue theorem we have $\oint_C f(z)dz = 2\pi i \operatorname{Res}\left(\frac{\pi}{2}\right)$

Now $\operatorname{Res}\left(\frac{\pi}{2}\right) = \dfrac{1}{(3-1)!}\lim_{z\to\frac{\pi}{2}}\dfrac{d^2}{dz^2}\left[(z-\frac{\pi}{2})^3 f(z)\right]$

$= \frac{1}{2}\lim_{z\to\frac{\pi}{2}}(-2\sin z - z\cos z)$ = -1

Therefore $\oint_C f(z)dz = 2\pi i(-1)$

Hence $\oint_C \dfrac{z\cos z}{\left(z-\frac{\pi}{2}\right)^3}dz = -2\pi i$.

Ex 2. Evaluate $\oint_C \dfrac{z+1}{z^2-2z}dz$ where C is the circle $|z| = 5$

Sol. Left as an exercise.

Module IV
(Partial Differential Equation (PDE) & Series Solution Of Ordinary Differential Equation (ODE))

LECTURE 35.

Basic Concepts of PDE

Origin of PDE

With the knowledge of functions of several variables and the concept of a partial derivative, one can generalize the concept of a differential equation to include equations that involve partial derivatives, not just ordinary ones. Solutions to such equations will involve functions not just of one variable, but of several variables. Such equations arise naturally, for example, when one is working with situations that involve positions in space that vary over time. To model such a situation, one needs to use functions that have several variables to keep track of the spatial dimensions and an additional variable for time.

Partial differential equations are ubiquitous in science, as they describe phenomena such as fluid flow, gravitational fields, and electromagnetic fields. They are important in fields such as aircraft simulation, computer graphics, and weather prediction. The central equations of general relativity and quantum mechanics are also partial differential equations.

Examples of some important PDEs

(1) $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ *One-dimensional wave equation*

(2) $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ *One-dimensional heat equation*

(3) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ *Two-dimensional Laplace equation*

(4) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f(x, y)$ *Two-dimensional Poisson equation*

Note that for PDEs one typically uses some other function letter such as u instead of y, which now quite often shows up as one of the variables involved in the multivariable function.

Order and Degree

In general we can use the same terminology to describe PDEs as in the case of ODEs. For starters, we will call any equation involving one or more partial derivatives of a multivariable function a partial differential equation. The order of such an equation is the highest order partial derivative that shows up in the equation. In addition, the equation is called linear if it is of the first degree in the unknown function u, and its partial derivatives, u_x, u_{xx}, u_y, etc. (this means that the highest power of the function, u, and its derivatives is just equal to one in each term in the equation, and that only one of them appears in each term). If each term in the equation involves either u, or one of its partial derivatives, then the function is classified as homogeneous.

Take a look at the list of PDEs above. Try to classify each one using the terminology given above. Note that the function in the Poisson equation is just a function of the variables x and y, it has nothing to do with .

Answers: All of these PDEs are second order, and are linear. All are also homogeneous except for the fourth one, the Poisson equation, as the term on the right hand side doesn't involve u or any of its derivatives.

The reason for defining the classifications linear and homogeneous for PDEs is to bring up the principle of superposition. This excellent principle (which also shows up in the study of linear homogeneous ODEs) is useful exactly whenever one considers solutions to linear homogeneous PDEs. The idea is that if one has two functions, u_1 and u_2 that satisfy a linear homogeneous differential equation, then since taking the derivative of a sum of functions is the same as taking the sum of their derivatives, then as long as the highest powers of derivatives involved in the equation are one (i.e., that it's linear), and that each term has a derivative in it (i.e. that it's homogeneous), then it's a straightforward exercise to see that the sum of u_1 and u_2 will also be a solution to the differential equation. In fact, so will any linear combination, $au_1 + bu_2$, where a and b are constants.

For instance, the two functions $\cos(xy)$ and $\sin(xy)$ are both solutions for the first-order linear homogeneous PDE:

(5) $$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0$$

It's a simple exercise to check that $\cos(xy) + \sin(xy)$ and $3\cos(xy) - 2\sin(xy)$ are also solutions to the same PDE (as will be any linear combination of $\cos(xy)$ and $\sin(xy)$)

This principle is extremely important, as it enables us to build up particular solutions out of infinite families of solutions through the use of Fourier series.

Solution PDEs

Solving PDEs is considerably more difficult in general than solving ODEs, as the level of complexity involved can be great. For instance the following seemingly completely unrelated functions are all solutions to the two-dimensional Laplace equation:

(1) $x^2 - y^2$, $e^x \cos(y)$ and $\ln(x^2 + y^2)$

You should check to see that these are all in fact solutions to the Laplace equation by doing the same thing you would do for an ODE solution, namely, calculate $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$, substitute them into the PDE equation and see if the two sides of the equation are identical.

Now, there are certain types of PDEs for which finding the solutions is not too hard. For instance, consider the first-order PDE

(2) $$\frac{\partial u}{\partial x} = 3x^2 + xy^2$$

where *u* is assumed to be a two-variable function depending on *x* and *y*. How could you solve this PDE? Think about it, is there any reason that we couldn't just undo the partial derivative of *u* with respect to *x* by integrating with respect to *x*? No, so try it out! Here, note that we are given information about just one of the partial derivatives, so when we find a solution, there will be an unknown factor that's not necessarily just an arbitrary *constant*, but in fact is a completely arbitrary *function* depending on *y*.

To solve (2), then, integrate both sides of the equation with respect to *x*, as mentioned. Thus

(3) $$\int \frac{\partial u}{\partial x}\,dx = \int (3x^2 + xy^2)\,dx$$

so that $u(x,y)=x^3+\frac{1}{2}x^2y^2+F$. What is *F*? Note that it could be any function such that when one takes its partial derivative with respect to *x*, the result is 0. This means that in the case of PDEs, the arbitrary constants that we ran into during the course of solving ODEs are now taking the form of whole functions. Here *F*, is in fact any function, *F(y),* of *y* alone. To check that this is indeed a solution to the original PDE, it is easy enough to take the partial derivative of this $u(x,y)$ function and see that it indeed satisfies the PDE in (2).

Now consider a second-order PDE such as

(4) $$\frac{\partial^2 u}{\partial x \partial y}=5x+y^2$$

where *u* is again a two-variable function depending on *x* and *y*. We can solve this PDE by integrating first with respect to *x*, to get to an intermediate PDE,

(5) $$\frac{\partial u}{\partial y}=\frac{5}{2}x^2+xy^2+F(y)$$

where *F(y)* is a function of *y* alone. Now, integrating both sides with respect to *y* yields

(6) $$u(x,y)=\frac{5}{2}x^2y+\frac{1}{3}xy^3+F(y)+G(x)$$

where now *G(x)* is a function of *x* alone (Note that we could have integrated with respect to *y* first, then *x* and we would have ended up with the same result). Thus, whereas in the ODE world, general solutions typically end up with as many arbitrary *constants* as the order of the original ODE, here in the PDE world, one typically ends up with as many arbitrary *functions* in the general solutions.

To end up with a specific solution, then, we will need to be given extra conditions that indicate what these arbitrary functions are. Thus the initial conditions for PDEs will typically involve knowing whole functions, not just constant values. We will also see that the initial conditions that appeared in specific ODE situations have slightly more involved analogs in the PDE world, namely there are often so-called *boundary* conditions as well as initial conditions to take into consideration.

Introduction to different methods of solution of PDEs

Linear PDEs are generally solved, when possible, by decomposing the equation according to a set of basis functions, solving those individually and using superposition to find the solution corresponding to the boundary conditions. The method of separation of variables has many important particular applications.

There are no generally applicable methods to solve non-linear PDEs. Still, existence and uniqueness results (such as the Cauchy-Kovalevskaya theorem) are often possible, as are proofs of important qualitative and quantitative properties of solutions (getting these results is a major part of analysis).

Nevertheless, some techniques can be used for several types of equations. The h-principle is the most powerful method to solve underdetermined equations. The Riquier-Janet theory is an effective method for obtaining information about many analytic over-determined systems.

The method of characteristics can be used in some very special cases to solve partial differential equations.

In some cases, a PDE can be solved via perturbation analysis in which the solution is considered to be a correction to an equation with a known solution. Alternatives are numerical analysis techniques ranging from finite difference schemes to multi-grid, finite element and finite volume methods. Many interesting problems in science and engineering are solved in this way using computers, sometimes high performance supercomputers. However, most problems in science and engineering are tackled using scientific computing rather than numerical analysis, as usually it is not known whether the numerical methods used produce solutions close to the true ones.

Classification

Second-order partial differential equations, and systems of second-order PDEs, can usually be classified as parabolic, hyperbolic or elliptic. This classification gives an intuitive insight into the behaviour of the system itself. The general second-order PDE is of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + \cdots = 0,$$

which looks remarkably similar to the equation for a conic section:

$$Ax^2 + 2Bxy + Cy^2 + \cdots = 0.$$

The reason B has a coefficient of 2 is due to the assumed commutativity of partial derivatives in the first case (recall that mixed derivatives which are continuous do not depend on the order of taking the partial derivatives in the different variables!) , and the commutativity of multiplication in the second. Just as one classifies conic sections into parabolic, hyperbolic, and elliptic based on the discriminant $B^2 - AC$, the same can be done for a second-order PDE.

$B^2 - AC < 0$: elliptic equations tend to smooth out any disturbances. A typical example is Laplace's equation. The motion of a fluid at sub-sonic speeds can be approximated with elliptic PDEs.

$B^2 - AC = 0$: parabolic equations tend to smooth out any pre-existing disturbances in the data. A typical example is the heat equation.

$B^2 - AC > 0$: hyperbolic equations tend to amplify any disturbances in the data. A typical example is the wave equation. The motion of a fluid at super-sonic speeds can be approximated with hyperbolic PDEs.

This method of classification can easily be extended to systems of partial differential equations by examining the eigenvalues of the coefficient matrix. In this situation, the classification scheme becomes:

Elliptic: The eigenvalues are all positive or all negative.

Parabolic : The eigenvalues are all positive or all negative, save one which is zero.

Hyperbolic : There is at least one negative and at least one positive eigenvalue, and none of the eigenvalues are zero.

This matches with positive-definite and negative-definite matrix analysis, of the sort that comes up during a discussion of maxima and minima. Moreover, using the concepts of positive-definiteness and negative-definiteness, it is possible to extend this classification to PDEs and systems of PDEs which are of order higher than 2 (as well as for systems of PDEs of 1^{st} order).

LECTURE 36.

Separation of Variable method

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There are several approaches to solve PDEs. The first one we will work with, using a technique called separation of variables, again, demonstrates one of the most widely used solution techniques for PDEs. The idea behind it is to split up the original PDE into a series of simpler ODEs, each of which we should be able to solve readily using tricks already learned. The second technique, which we will see in the next section, uses a transformation trick that also reduces the complexity of the original PDE, but in a very different manner. This second solution is due to Jean Le Rond D'Alembert (an 18th century French mathematician), and is called D'Alembert's solution, as a result.

First, note that for a specific wave equation situation, in addition to the actual PDE, we will also have boundary conditions arising from the fact that the endpoints of the string are attached solidly, at the left end of the string, when $x = 0$ and at the other end of the string, which we suppose has overall length l. Let's start the process of solving the PDE by first figuring out what these boundary conditions imply for the solution function, $u(x,t)$.

Answer: for all values of t, the time variable, it must be the case that the vertical displacement at the endpoints is 0, since they don't move up and down at all, so that

(1) $u(0,t)=0$ and $u(l,t)=0$ for all values of t

are the **boundary conditions** for our wave equation. These will be key when we later on need to sort through possible solution functions for functions that satisfy our particular vibrating string set-up.

You might also note that we probably need to specify what the shape of the string is right when time $t = 0$, and you're right - to come up with a particular solution function, we would need to know $u(x,0)$. In fact we would also need to know the initial velocity of the string, which is just $u_t(x,0)$. These two requirements are called the **initial conditions** for the wave equation, and are also necessary to specify a particular vibrating string solution. For instance, as the simplest example of initial conditions, if no one is plucking the string, and it's perfectly flat to start with, then the initial conditions would just be $u(x,0)=0$ (a perfectly flat string) with initial velocity, $u_t(x,0)=0$. Here, then, the solution function is pretty unenlightening – it's just $u(x,t)=0$, i.e. no movement of the string through time.

To start the separation of variables technique we make the key assumption that whatever the solution function is, that it can be written as the product of two independent functions, each one of which depends on just one of the two variables, x or t. Thus, imagine that the solution function, $u(x,t)$ can be written as

(2) $u(x,t)=F(x)G(t)$

where F, and G, are single variable functions of x and t respectively. Differentiating this equation for $u(x,t)$ twice with respect to each variable yields

(3) $$\frac{\partial^2 u}{\partial x^2} = F''(x)G(t) \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} = F(x)G''(t)$$

LECTURE 37.

One-Dimensional Wave Equation

Thus when we substitute these two equations back into the original wave equation, which is

(4) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

then we get

(5) $$\frac{\partial^2 u}{\partial t^2} = F(x)G''(t) = c^2 \frac{\partial^2 u}{\partial x^2} = c^2 F''(x)G(t)$$

Here's where our separation of variables assumption pays off, because now if we separate the equation above so that the terms involving F and its second derivative are on one side, and likewise the terms involving G and its derivative are on the other, then we get

(6) $$\frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)}$$

Now we have an equality where the left-hand side just depends on the variable t, and the right-hand side just depends on x. Here comes the critical observation - how can two functions, one just depending on t, and one just on x, be equal for all possible values of t and x? The answer is that they must each be constant, for otherwise the equality could not possibly hold for all possible combinations of t and x. Thus we have

(7) $$\frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = k$$

where k is a constant. First let's examine the possible cases for k.

Case One: k = 0

Suppose k equals 0. Then the equations in (7) can be rewritten as

(8) $G''(t) = 0 \cdot c^2 G(t) = 0$ and $F''(x) = 0 \cdot F(x) = 0$

yielding with very little effort two solution functions for F and G:

(9) $G(t) = at + b$ and $F(x) = px + r$

where a,b, p and r, are constants (note how easy it is to solve such simple ODEs versus trying to deal with two variables at once, hence the power of the separation of variables approach).

Putting these back together to form $u(x,t) = F(x)G(t)$, then the next thing we need to do is to note what the boundary conditions from equation (1) force upon us, namely that

(10) $u(0,t) = F(0)G(t) = 0$ and $u(l,t) = F(l)G(t) = 0$ for all values of t

Unless $G(t) = 0$ (which would then mean that $u(x,t) = 0$, giving us the very dull solution equivalent to a flat, unplucked string) then this implies that

(11) $F(0) = F(l) = 0$.

But how can a linear function have two roots? Only by being identically equal to 0, thus it must be the case that $F(x) = 0$. Sigh, then we still get that $u(x,t) = 0$, and we end up with the dull solution again, the only possible solution if we start with $k = 0$.

So, let's see what happens if…

Case Two: k > 0

So now if k is positive, then from equation (7) we again start with

(12) $G''(t) = kc^2 G(t)$

and

(13) $F''(x) = kF(x)$

Try to solve these two ordinary differential equations. You are looking for functions whose second derivatives give back the original function, multiplied by a positive constant. Possible candidate solutions to consider include the exponential and sine and cosine functions. Of course, the sine and cosine functions don't work here, as their second derivatives are *negative* the original function, so we are left with the exponential functions.

Let's take a look at (13) more closely first, as we already know that the boundary conditions imply conditions specifically for $F(x)$, i.e. the conditions in (11). Solutions for $F(x)$ include anything of the form

(14) $F(x) = Ae^{\omega x}$

where $\omega^2 = k$ and A is a constant. Since ω could be positive or negative, and since solutions to (13) can be added together to form more solutions (note (13) is an example of a second order linear homogeneous ODE, so that the superposition principle holds), then the general solution for (13) is

(14) $F(x) = Ae^{\omega x} + Be^{-\omega x}$

where now A and B are constants and $\omega = \sqrt{k}$. Knowing that $F(0) = F(l) = 0$, then unfortunately the only possible values of A and B that work are $A = B = 0$, i.e. that $F(x) = 0$. Thus, once again we end up with $u(x,t) = F(x)G(t) = 0 \cdot G(t) = 0$, i.e. the dull solution once more. Now we place all of our hope on the third and final possibility for *k*, namely…

Case Three: k < 0

So now we go back to equations (12) and (13) again, but now working with *k* as a negative constant. So, again we have

(12) $G''(t) = kc^2 G(t)$

and

(13) $F''(x) = kF(x)$

Exponential functions won't satisfy these two ODEs, but now the sine and cosine functions will. The general solution function for (13) is now

(15) $F(x) = A\cos(\omega x) + B\sin(\omega x)$

where again *A* and *B* are constants and now we have $\omega^2 = -k$. Again, we consider the boundary conditions that specified that $F(0) = F(l) = 0$. Substituting in 0 for *x* in (15) leads to

(16) $F(0) = A\cos(0) + B\sin(0) = A = 0$

so that $F(x) = B\sin(\omega x)$. Next, consider $F(l) = B\sin(\omega l) = 0$. We can assume that *B* isn't equal to 0, otherwise $F(x) = 0$ which would mean that $u(x,t) = F(x)G(t) = 0 \cdot G(t) = 0$, again, the trivial unplucked string solution. With $B \neq 0$, then it must be the case that $\sin(\omega l) = 0$ in order to have $B\sin(\omega l) = 0$. The only way that this can happen is for ωl to be a multiple of π. This means that

(17) $\omega l = n\pi$ or $\omega = \dfrac{n\pi}{l}$ (where *n* is an integer)

This means that there is an infinite set of solutions to consider (letting the constant B be equal to 1 for now), one for each possible integer n.

(18) $$F(x) = \sin\left(\frac{n\pi}{l}x\right)$$

Well, we would be done at this point, except that the solution function $u(x,t) = F(x)G(t)$ and we've neglected to figure out what the other function, $G(t)$, equals. So, we return to the ODE in (12):

(12) $$G''(t) = kc^2G(t)$$

where, again, we are working with k, a negative number. From the solution for $F(x)$ we have determined that the only possible values that end up leading to non-trivial solutions are with $k = -\omega^2 = -\left(\frac{n\pi}{l}\right)^2$ for n some integer. Again, we get an infinite set of solutions for (12) that can be written in the form

(19) $$G(t) = C\cos(\lambda_n t) + D\sin(\lambda_n t)$$

where C and D are constants and $\lambda_n = c\sqrt{-k} = c\omega = \frac{cn\pi}{l}$, where n is the same integer that showed up in the solution for $F(x)$ in (18) (we're labeling λ with a subscript "n" to identify which value of n is used).

Now we really are done, for all we have to do is to drop our solutions for $F(x)$ and $G(t)$ into $u(x,t) = F(x)G(t)$, and the result is

(20) $$u_n(x,t) = F(x)G(t) = \left(C\cos(\lambda_n t) + D\sin(\lambda_n t)\right)\sin\left(\frac{n\pi}{l}x\right)$$

where the integer n that was used is identified by the subscript in $u_n(x,t)$ and λ_n, and C and D are arbitrary constants.

LECTURE 38.

One-Dimensional Heat Equation

We simplify our heat diffusion modeling by considering the specific case of heat flowing in a long thin bar or wire, where the cross-section is very small, and constant, and insulated in such a way that the heat flow is just along the length of the bar or wire. In this slightly contrived situation, we can model the heat flow by keeping track of the temperature at any point along the bar using just one spatial dimension, measuring the position along the bar.

This means that the function, u, that keeps track of the temperature, just depends on x, the position along the bar, and t, time, and so the heat equation from the previous section becomes the so-called **one-dimensional heat equation**:

(1) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

One of the interesting things to note at this point is how similar this PDE appears to the wave equation PDE. However, the resulting solution functions are remarkably different in nature. Remember that the solutions to the wave equation had to do with oscillations, dealing with vibrating strings and all that. Here the solutions to the heat equation deal with temperature flow, not oscillation, so that means the solution functions will likely look quite different. If you're familiar with the solution to Newton's heating and cooling differential equations, then you might expect to see some type of exponential decay function as part of the solution function.

Before we start to solve this equation, let's mention a few more conditions that we will need to know to nail down a specific solution. If the metal bar that we're studying has a specific length, l, then we need to know the temperatures at the ends of the bars. These temperatures will give us boundary conditions similar to the ones we worked with for the wave equation. To make life a bit simpler for us as we solve the heat equation, let's start with the case when the ends of the bar, at $x = 0$ and $x = l$ both have temperature equal to 0 for all time (you can picture this situation as a metal bar with the ends stuck against blocks of ice, or some other cooling apparatus keeping the ends exactly at 0 degrees). Thus we will be working with the same boundary conditions as before, namely

(2) $u(0,t) = 0$ and $u(l,t) = 0$ for all values of t

Finally, to pick out a particular solution, we also need to know the initial starting temperature of the entire bar, namely we need to know the function $u(x,0)$. Interestingly, that's all we would need for an initial condition this time around (recall that to specify a particular solution in the wave equation we needed to know two initial conditions, $u(x,0)$ and $u_t(x,0)$).

The nice thing now is that since we have already solved a PDE, then we can try following the same basic approach as the one we used to solve the last PDE, namely separation of variables. With any luck, we will end up solving this new PDE. So, remembering back to what we did in that case, let's start by writing

(3) $$u(x,t) = F(x)G(t)$$

where *F*, and *G*, are single variable functions. Differentiating this equation for $u(x,t)$ with respect to each variable yields

(4) $$\frac{\partial^2 u}{\partial x^2} = F''(x)G(t) \quad \text{and} \quad \frac{\partial u}{\partial t} = F(x)G'(t)$$

When we substitute these two equations back into the original heat equation

(5) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

we get

(6) $$\frac{\partial u}{\partial t} = F(x)G'(t) = c^2 \frac{\partial^2 u}{\partial x^2} = c^2 F''(x)G(t)$$

If we now separate the two functions *F* and *G* by dividing through both sides, then we get

(7) $$\frac{G'(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)}$$

Just as before, the left-hand side only depends on the variable *t*, and the right-hand side just depends on *x*. As a result, to have these two be equal can only mean one thing, that they are both equal to the same constant, *k*:

(8) $$\frac{G'(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = k$$

As before, let's first take a look at the implications for $F(x)$ as the boundary conditions will again limit the possible solution functions. From (8) we get that $F(x)$ has to satisfy

(9) $$F''(x) - kF(x) = 0$$

Just as before, one can consider the various cases with *k* being positive, zero, or negative. Just as before, to meet the boundary conditions, it turns out that *k* must in fact be negative (otherwise

$F(x)$ ends up being identically equal to 0, and we end up with the trivial solution $u(x,t)=0$). So skipping ahead a bit, let's assume we have figured out that *k* must be negative (you should check the other two cases just as before to see that what we've just written is true!). To indicate this, we write, as before, that $k=-\omega^2$, so that we now need to look for solutions to

(10) $\quad F''(x)+\omega^2 F(x)=0$

These solutions are just the same as before, namely the general solution is:

(11) $\quad F(x)=A\cos(\omega x)+B\sin(\omega x)$

where again *A* and *B* are constants and now we have $\omega=\sqrt{-k}$. Next, let's consider the boundary conditions $u(0,t)=0$ and $u(l,t)=0$. These are equivalent to stating that $F(0)=F(l)=0$. Substituting in 0 for *x* in (11) leads to

(12) $\quad F(0)=A\cos(0)+B\sin(0)=A=0$

so that $F(x)=B\sin(\omega x)$. Next, consider $F(l)=B\sin(\omega l)=0$. As before, we check that *B* can't equal 0, otherwise $F(x)=0$ which would then mean that $u(x,t)=F(x)G(t)=0\cdot G(t)=0$, the trivial solution, again. With $B\neq 0$, then it must be the case that $\sin(\omega l)=0$ in order to have $B\sin(\omega l)=0$. Again, the only way that this can happen is for ωl to be a multiple of π. This means that once again

(13) $\quad \omega l=n\pi$ or $\omega=\dfrac{n\pi}{l}$ (where *n* is an integer)

and so

(14) $\quad F(x)=\sin\left(\dfrac{n\pi}{l}x\right)$

where *n* is an integer. Next we solve for $G(t)$, using equation (8) again. So, rewriting (8), we see that this time

(15) $\quad G'(t)+\lambda_n{}^2 G(t)=0$

where $\lambda_n=\dfrac{cn\pi}{l}$, since we had originally written $k=-\omega^2$, and we just determined that $\omega=\dfrac{n\pi}{l}$ during the solution for $F(x)$. The general solution to this first order differential equation is just

(16) $G(t) = Ce^{-\lambda_n{}^2 t}$

So, now we can put it all together to find out that

(17) $u(x,t) = F(x)G(t) = C\sin\left(\frac{n\pi}{l}x\right)e^{-\lambda_n{}^2 t}$

where n is an integer, C is an arbitrary constant, and $\lambda_n = \frac{cn\pi}{l}$. As is always the case, given a supposed solution to a differential equation, you should check to see that this indeed is a solution to the original heat equation, and that it satisfies the two boundary conditions we started with.

LECTURE 39.

Two-Dimensional Laplace Equation

We want to solve the Dirichlet boundary value problem for the Laplace equation in a rectangle

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} & = 0, \quad 0 < x < L, \quad 0 < y < H, \\ u(0, y) & = g_1(y), \\ u(L, y) & = g_2(y), \\ u(x, 0) & = g_3(x), \\ u(x, H) & = g_4(x). \end{cases} \tag{1}$$

By superposition, we know that we can split problem (1) into four similar problems, each of which satisfies one of the four boundary functions and the zero boundary condition on the other three sides. For example, let us find u_1 :

$$\begin{cases} \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} & = 0, \\ u_1(0, y) & = g_1(y), \\ u_1(L, y) & = 0, \\ u_1(x, 0) & = 0, \\ u_1(x, H) & = 0. \end{cases} \tag{2}$$

We use the method of **separation of variables**. Let $u_1(x, y) = X(x)Y(y)$. Then the Laplace equation can be written as

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}. \tag{3}$$

Since the two sides of (3) are functions of different variables, we conclude that they must be constant, which we set to be λ. Thus (3) becomes

$$X'' = \lambda X, \tag{4}$$

$$Y'' = -\lambda Y. \tag{5}$$

We see that we can let $Y(y)$ satisfy the boundary condition

$$Y(0) = Y(H) = 0. \tag{6}$$

Then, the Y equation (5) and the boundary condition (6) have solution

$$Y = c\sin(\frac{n\pi y}{H}) \tag{7}$$

for

$$\lambda = (\frac{n\pi}{H})^2, \tag{8}$$

where $n = 1, 2, \cdots$. For the λ in (8), we find that the general solution to (4) is

$$X(x) = Ee^{\frac{n\pi}{H}x} + Fe^{-\frac{n\pi}{H}x}.$$

Or

$$X(x) = a_1\cosh[\frac{n\pi}{H}(x-L)] + a_2\sinh[\frac{n\pi}{H}(x-L)].$$

The shift in x by L is selected to satisfy the boundary condition at $x = L$ conveniently. We impose that $X(L) = 0$, which implies $a_1 = 0$. Thus

$$X(x) = a_2\sinh[\frac{n\pi}{H}(x-L)].$$

In summary, solutions u of the product form $X(x)Y(y)$ satisfying the zero condition on the three corresponding sides of (2) are

$$u(x,y) = a_2\sinh[\frac{n\pi}{H}(x-L)]\sin(\frac{n\pi y}{H})$$

for any constant a_2 and all $n = 1, 2, 3, \cdots$. By superposition, we find that

$$u(x,y) = \sum_{n=1}^{\infty} a_n\sinh[\frac{n\pi}{H}(x-L)]\sin(\frac{n\pi y}{H}) \tag{9}$$

is also a solution to the Laplace equation with zero value on the three zero-value sides of the rectangle in (2) for any real numbers a_n. We want to choose a_n such that u in (9) satisfies the fourth nonzero boundary condition:

$$u(0,y) = \sum_{n=1}^{\infty} a_n\sinh[-\frac{n\pi L}{H}]\sin(\frac{n\pi y}{H}) = g_1(y).$$

LECTURE 40.

Power Series – Properties

Definition

A power series is the sum of the infinite number of terms of the form

$$S = a_o + a_1(x - x_o) + a_2(x - x_o)^2 + \dots = \sum_{m=0}^{\infty} a_m (x - x_o)^m$$

where a_o, a_1, a_2, … are constants, called the **coefficients** of the series. x_o is a constant, called the **center** of the series. A power series does not include terms with negative powers.

* The linear differential equations with constant coefficients always possess series solutions.

The homogeneous solution of the linear differential equation with constant coefficients

will have one of the type:

All of these solutions can be expanded in power series of x

$$e^x = 1 + x + \frac{x^2}{2!} + \dots = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$

The series power form of y can be accepted as a solution provided that the differential equation is satisfied by it and the series is convergent. Some differential equations with variable coefficients possess series solutions.

Some properties of power series

The n th partial sum of the series is

$$S_n(x) = a_o + a_1(x - x_o) + a_2(x - x_o)^2 + \dots + a_n(x - x_o)^n$$

and the remainder is

$$R_n(x) = a_{n+1}(x - x_o)^{n+1} + a_{n+2}(x - x_o)^{n+2} + \dots$$

A power series converges if $R_n \to 0$ as $n \to \infty$; otherwise, it diverges

There is usually an interval over which the power series converges with the center at $x = x_o$; that is, the series converges if

$$|x - x_o| < R$$

where R is called the radius of convergence. The radius of convergence can be obtained from

$$R = \lim (m \to \infty) \left| \frac{a_m}{a_{m+1}} \right|$$

EX: $$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^m}{m!} + \frac{x^{m+1}}{(m+1)!} + \dots$$

Radius of convergence

$$R = \lim (m \to \infty) \left| \frac{a_m}{a_{m+1}} \right| = \lim (m \to \infty) \left| \frac{\frac{1}{m!}}{\frac{1}{(m+1)!}} \right| = \lim (m \to \infty) \left| \frac{(m+1)!}{m!} \right| = \lim (m \to \infty) |m+1| = \infty$$

A function y(x) is analytic at the point $x = x_o$ if it can be expressed as a power series $\sum_{m=0}^{\infty} a_m (x - x_o)^m$ with $R > 0$.

If the functions p(x), q(x), and r(x) in the differential equation

$$y'' + p(x)y' + q(x)y = r(x)$$

are analytic at the point $x = x_o$, the solution can be represented by a power series with a finite radius of convergence, that is,

$$y(x) = \sum_{m=0}^{\infty} a_m (x - x_o)^m \quad \text{with } R > 0$$

The point $x = x_o$ is called an ordinary (or regular) point.

EX: $y'' = e^x y$, every point $x \neq \infty$ is a regular point

$x^5 y'' = y$, every point x except for $x = 0$ and $x = \infty$ is a regular point

If p(x), q(x), or r(x) is not analytic at $x = x_o$, the point $x = x_o$ is said to be a singular point

Regular singular point and Irregular singular point

Consider a second order homogeneous linear equation

$$y'' + p(x)y' + q(x)y = 0$$

Regular singular point: The point $x = x_o$ is called a regular singular point of (1) if not both of p(x), q(x) are analytic but both $(x - x_o)p(x)$ and $(x - x_o)^2 q(x)$ are analytic in the neighborhood of x_o.

Irregular singular point: The point $x = x_o$ is called an irregular singular point of (1) if it is neither a regular point nor a regular singular point.

EX:(a) $(x - 1)y'' = y$ has a regular singular point at 1

(b) $x^2 y'' + xy' = y$ has a regular singular point at 0

(c) $x^3 y'' = (x + 1)y$ has an irregular singular point at 0

If $x = x_o$ is a regular point of the differential equation then the power series method can be applied. The general solution of Eq. (4.1-1) is $y = Ay_1(x) + By_2(x)$ where y_1 and y_2 are linearly independent series solutions ($\sum_{m=0}^{\infty} a_m (x - x_o)^m$) which are analytic at $x = x_o$. The radius of convergence for each of the series solutions y_1 and y_2 is at least as large as the minimum of the radii of convergence of the series for p(x) and q(x).

LECTURE 41.

Series solution of ODE-I

Problems on series solution of ODE

Find the power series solution of the equation
in powers of .

Solution: .... (i)

Comparing with where and are polynomial in or constants, we have

Here, So, is an ordinary point of (1).

We take power series

(ii)

;

Substituting the above values of and in (1), we get

(iii)

Since (iii) is an identity ; equating the constant term and the coefficient of various power of to zero, we get

and

(iv)

(The above relation (iv) is known as recurrence relation.)

Putting in (iv)   as

Putting in (iv)

Putting the above values etc in (ii), we have

where and are arbitrary constants.

LECTURE 42.

Series solution of ODE-II

1. Use method of Frobenius to find the solution of the differential equation

.

Ans: Given,

(i)

Comparing (i) with where and are polynomial in or constants, we have

, so, is a singular point of (i)

Now,

and

and

Since and possess derivatives of all orders in the neighbourhood of . Hence is a regular singular point of (i).

Consider the solution of (i) in the form

, where (ii)

Substituting these values in (i), we get

.... (iii)

which is an identity in . Equating to zero the coefficient of the lowest power of , namely , then from (iii), we get the identical equation =0 as .

Next equating to zero the coefficient of , the recurrence relation is given by

.... (iv)

Putting in (iv), we get

and so on.

....

Substituting these values of in (ii), we get

Hence the required complete solution is

where and are arbitrary constants.

2. Obtain the power series solution of the differential equation:

Solution: Given,

Comparing (i) with where and are polynomial in or constants, we have .

So, is a singular part of (i).

Now,

and

and

Since and possess derivatives of all orders in the neighbourhood of . Hence is a regular singular point of (i).

Consider the solution of (i) in the form

, where

.... (ii)

Substituting these values in (i), we get

.... (iii)

which is an identity in . Equating to zero the coefficient of the lowest power of , namely , (iii) gives identity equation

Here the roots of the identity equation are unequal and do not differ by an integer.

Next equating to zero the coefficient of ,we get the recurrence relation as

.... (iv)

Putting in (iv), we get

and so on.

Substituting these values of in (ii), we get

and
Hence the required complete solution is

where and are arbitrary constants.

LECTURE 43.

Bessel's Function

Definition

The differential equation of the form

is called Bessel's equation of order n, n being a non-homogeneous constant.

Solution of Bessel's Equation

The general solution of Bessel's equation is

where &

A,B are arbitrary constants.

Recurrence formulae

1. or

Proof: We know that

=

(proved)

Some other recurrence relation

or

Problems of Bessel's Equation

1. Show that when n is a positive integer

i.

ii. for positive or negative integers.

Solution**:** (i) we have

Since if p is an integer ,then is infinity for .so we get terms in equal to zero till i.e

Hence we can write

by putting

(ii) Left as an exercise.

2. Show that cos x

sin x

Solution:

(i) We know that

= (1)

Putting n in (1) ,we get

=

(ii) Left as an exercise.

LECTURE 44.

Legendre Polynomial

Definition

The differential equation of the form

.... (i)

Is called Legendre's equation where n is a positive integer.

This equation can also be written as

General solution of Legendre's equation

The general solution of Legendre's equation is

where and re arbitrary constants.

where

and

Theorem: is the coefficient of in the expansion of in ascending powers of where

Orthogonal properties of Legendre's polynomials

(i)

(ii)

Proof: (i) The Legendre's equation may be written as

.... (i)

and .... (ii)

Multiplying (i) by and (ii) by and then subtracting, we have

or,

Integrating between the limits to we have,

(ii) we have,

Squaring both sides, we have

Integrating between the limits to we have,

or, [since other integral on L.H.S. is zero, as m]

Equating the coefficient of , we have

.

Recurrence formula

(i)
(ii)
(iii)
(iv)

Proof: (i) we have,

Differentiating both side w.r.t. we have

or,

or,

equating the coefficients of from both sides, we have

(ii), (iii), (iv) left as exercise.

Exercise

(1) Express in terms of Legendre's polynomials.

(2) Show that,

(i)

(ii)